PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2012 Series)

Problem:

If x is a positive number, and if n is a positive integer, show that

$$(1+x)(1+x^2)(1+x^3)\cdots(1+x^n) \geq (1+x^{\frac{n+1}{2}})^n.$$

Solution: (by Bennett Marsh, Sophomore, ECE, Purdue University)

First, observe that for all numbers u and v, $u^2+v^2 \geq 2uv$, since

$$u^2 - 2uv + v^2 = (u-v)^2 \geq 0.$$

If n is even, then we can rearrange terms to get

$$(1+x)(1+x^n)\cdot(1+x^2)(1+x^{n-1})\cdots(1+x^{\frac{n}{2}})(1+x^{\frac{n+1}{2}}),$$

so that there are $\dfrac{n}{2}$ terms multiplied together, each of the form $(1+x^a)(1+x^b)$, with $a+b=n+1$. Using the above result, we have

$$\begin{aligned}(1+x^a)(1+x^b) &= 1 + x^a + x^b + x^{a+b}\\ &\geq 1 + 2x^{\frac{a}{2}}x^{\frac{b}{2}} + x^{a+b}\\ &= (1+x^{\frac{a+b}{2}})^2\\ &= (1+x^{\frac{n+1}{2}})^2.\end{aligned}$$

Thus,

$$(1+x)(1+x^2)\cdots(1+x^n) \geq ((1+x^{\frac{n+1}{2}})^2)^{\frac{n}{2}} = (1+x^{\frac{n+1}{2}})^n.$$

If n is odd, then we can form $\dfrac{n-1}{2}$ such pairs, with an extra term of $(1+x^{\frac{n+1}{2}})$ left over. Then

$$(1+x)(1+x^2)\cdots(1+x^n) \geq ((1+x^{\frac{n+1}{2}})^2)^{\frac{n-1}{2}}(1+x^{\frac{n+1}{2}}) = (1+x^{\frac{n+1}{2}})^n.$$

The problem was also solved by:

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