PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2012 Series)

Problem:

For every integer $n > 2$ let $L(n)$ denote the sum of the integers from 1 through $[n/2]$ which are relatively prime to n, and let $U(n)$ denote the sum of the integers from $[n/2]+1$ through n which are relatively prime to n. Prove that if n is divisible by 4, then $U(n)/L(n) = 3$. ([ ] is the greatest integer function.)

**This problem was proposed by Steve Spindler. We also belatedly note that problem 10 of the Fall 2011 series was contributed by Hubert Desprez.

Solution: (by Pierre Castelli, Math Teacher, Antibes, France)

Let $n > 2$ be an integer divisible by $4 : n = 4m$.
Let j be a positive integer. Since $\gcd(j,n) = 1 \iff \gcd(j,2m) = 1$ we can write:

$$L(n) = \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} j \qquad \text{and} \qquad U(n) = \sum_{\substack{2m+1 \le j \le 4m-1 \\ \gcd(j,2m)=1}} j.$$

For $1 \le j \le 2m-1, \gcd(j,2m) = 1 \iff \gcd(2m-j,2m) = 1$, thus:

$$2L(n) = \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} j + \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} (2m-j) = \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} 2m.$$

For $1 \le j \le 2m-1, \gcd(j,2m) = 1 \iff \gcd(2m+j,2m) = 1$, so:

$$U(n) = \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} (2m+j) = \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} 2m + \sum_{\substack{1 \le j \le 2m-1 \\ \gcd(j,2m)=1}} j = 2L(n) + L(n) = 3L(n).$$

Finally:

$$\frac{U(n)}{L(n)} = 3.$$

The problem was also solved by:

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