PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2012 Series)

Problem:

What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \leq e \leq \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

The answer is $a_{\max} = \dfrac{1}{\ln 2} - 1$ and $b_{\min} = \dfrac{1}{2}$. Consider the functions $f, g, h : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{\ln(1+1/x)} - x$ with $f(1) = \dfrac{1}{\ln 2} - 1 > 0$. Applying L'Hospital twice we have

$$\begin{aligned}\lim_{x\to\infty} f(x)^{L'Hospital} &= \lim_{x\to\infty} \frac{-\ln(1+1/x) + \frac{1}{x+1} L'Hospital}{\frac{1}{x+1} - \frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}}{\frac{1}{x^2} - \frac{1}{(x+1)^2}} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x} + \frac{1}{x+1}} = \frac{1}{2}.\end{aligned}$$

Now we prove that f is increasing. $f'(x) = \dfrac{g(x)}{(\ln(1+1/x))^2}$ where $g(x) = \dfrac{1}{x} - \dfrac{1}{x+1} - (\ln(1+1/x))^2$. $g'(x) = \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) h(x)$ where $h(x) = 2\ln(1+1/x) - \dfrac{1}{x+1} - \dfrac{1}{x}$ and $h'(x) = \left(\dfrac{1}{x+1} - \dfrac{1}{x}\right)^2 > 0$. Therefore h is increasing, $\lim_{x\to\infty} h(x) = 0$ and so $h < 0$. Therefore $g' < 0$, g is decreasing, $\lim_{x\to\infty} g(x) = 0$ and so $g > 0$. Therefore $f' > 0$, and so f is increasing. Hence $f(1) \leq f(x) < \dfrac{1}{2}$ so $\ln\left(1+\dfrac{1}{x}\right)^{x+\frac{1}{\ln 2}-1} \leq 1 < \ln\left(1+\dfrac{1}{x}\right)^{x+\frac{1}{2}}$ and so for any $n \in \mathbb{N}^*$, $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{\ln 2}-1} \leq e < \left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}$. Note that $b_{\min} = \dfrac{1}{2}$ is optimal but there is no n such that the equality holds in the right side of the double inequality.

The problem was also solved by:

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