PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2012 Series)

Problem: Let M be the maximum of the numbers $f(k)$ for k an integer in $[0, 605]$, where

$$f(k) = \binom{605}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{605-k}.$$

Find all the integers in $[0, 605]$ satisfying $f(k) = M$.
Do not use a computer or tables.

Solution: (by Bennett Marsh, Sophomore, ECE, Purdue University)

For $f(k)$ to be a maximum, it is necessary (if $k < 605$) that $f(k) \geq f(k+1)$. Thus,

$$\frac{f(k)}{f(k+1)} = \frac{\frac{605!}{k!(605-k)!}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{605-k}}{\frac{605!}{(k!+1)(605-k-1)!}\left(\frac{1}{6}\right)^{k+1}\left(\frac{5}{6}\right)^{605-k-1}} = 5\cdot\frac{k+1}{605-k} \geq 1.$$

Solving, we find that $k \geq 100$. Similarly, at the maximum, $f(k) \geq f(k-1)$, and

$$\frac{f(k)}{f(k-1)} = \frac{1}{5}\cdot\frac{605-k+1}{k} \geq 1.$$

This leads to $k \leq 101$. Putting these two results together, we find that $100 \leq k \leq 101$. Thus, the maximum value $f(k) = M$ is achieved at both $k = 100$ and $k = 101$.

The problem was also solved by:

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