PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2012 Series)

Problem:

(a) Prove that if f is continuous on $[0,1]$ and differentiable on $(0,1)$ and satisfies $|f'(x)| \leq M$ for some positive number M then

$$\left| \int_0^1 f(x)dx - \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n} \right| \leq \frac{M}{n}$$

(b) (optional; the problem will be counted as solved if part (a) is solved) Show that the $\frac{M}{n}$ of part (a) can be improved to $\frac{M}{2n}$.

Solution: (by Yicun Qian, Junior, Math & Stat, Purdue University)

Let $a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx - f(\frac{k}{n})\frac{1}{n} \right|$ for $k = 0, 1, 2, \ldots, n-1$, it follows that $a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[f(x) - f(\frac{k}{n}) \right] dx \right|$. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, according to mean value theorem, there exists a point p in $\left(\frac{k}{n}, x\right)$ such that $f'(p) = \frac{f(x) - f(\frac{k}{n})}{x - \frac{k}{n}}$, for $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$. Since $|f'(x)| \leq M$ for all $x \in (0,1)$, we have $|f'(p)| = \left| \frac{f(x) - f(\frac{k}{n})}{x - \frac{k}{n}} \right| \leq M$, so $\left| f(x) - f(\frac{k}{n}) \right| \leq M. \left| x - \frac{k}{n} \right| = M \cdot \left(x - \frac{k}{n} \right)$ since $x > \frac{k}{n}$.

Hence

$$a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[f(x) - f(\frac{k}{n}) \right] dx \right| \leq \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} M \cdot \left(x - \frac{k}{n} \right) dx \right| = \left| \frac{1}{2}Mx^2 - \frac{Mk}{n}x \Big]_{\frac{k}{n}}^{\frac{k+1}{n}} \right| = \frac{M}{2n^2}.$$

And it turns out that

$$
\begin{aligned}
&\left| \int_0^1 f(x)dx - \sum_{k=0}^{n-1} f(\frac{k}{n})\frac{1}{n} \right| \\
&= \left| \int_0^{\frac{1}{n}} f(x)dx + \int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx + \cdots + \int_{\frac{n-1}{n}}^{1} f(x)dx - \left[f(0)\frac{1}{n} + f(\frac{1}{n})\frac{1}{n} + \cdots + f(\frac{n-1}{n})\frac{1}{n} \right] \right| \\
&= \left| \left(\int_0^{\frac{1}{n}} f(x)dx - f(0)\frac{1}{n} \right) + \left(\int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx - f(\frac{1}{n})\frac{1}{n} \right) + \cdots + \left(\int_{\frac{n-1}{n}}^{1} f(x)dx - f(\frac{n-1}{n})\frac{1}{n} \right) \right| \\
&\leq \left| \int_0^{\frac{1}{n}} f(x)dx - f(0)\frac{1}{n} \right| + \left| \int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx - f(\frac{1}{n})\frac{1}{n} \right| + \cdots + \left| \int_{\frac{n-1}{n}}^{1} f(x)dx - f(\frac{n-1}{n})\frac{1}{n} \right| \\
&= a_0 + a_1 + \cdots + a_{n-1} \leq \underbrace{\frac{M}{2n^2} + \frac{M}{2n^2} + \cdots + \frac{M}{2n^2}}_{n \text{ terms}} \\
&= \frac{M}{2n^2} \cdot n = \frac{M}{2n}
\end{aligned}
$$

The problem was also solved by:

<u>Undergraduates</u>: Kilian Cooley (Jr. Math & AAE), Bennett Marsh (So. Engr.), Scott Podlogar (Fr. Engr), Lirong Yuan (Jr. Math & CS)

<u>Graduates</u>: Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

<u>Others</u>: Manuel Barbero (New York), Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Sainyam Galhotra (CS student, IIT Delhi, India), Howard Heaton (Student, Computer Engr, Walla Walla Univ.), Scott Huber (Phys. Graduate Student, Ohio State Univ.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Parviz Khalili (Faculty, Christopher Newport Univ. VA), Anastasios Kotronis (Athens, Greece), Steven Landy (Physics Faculty, IUPUI), Kevin Laster (Indianapolis, IN), Matthew Lim, Boughanmi Mohamed Hedi(Tunisia), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Sun Hong Rhie(Granger, IN), Charles Roldan (BS Math 2010, Purdue), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Scoala Satulng, Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Devis Moises Alvarado Zavala(UPR)