PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2012 Series)

Problem:

Let I_k, $1 \le k \le n$, be intervals contained in $(0,1)$, and suppose the sum of the lengths of these intervals is 17. Show that there is a number in $(0,1)$ which is in at least five of the I_k.

Solution 1: (by Hubert Desprez, Paris, France)

Define $f_k(x) = \begin{cases} 1 & \text{if } x \in I_k, 1 \le k \le n \\ 0 & \text{else} \end{cases}$, $F = \sum_k f_k$. If each $x \in I = (0,1)$ is in at most four I_k's, then $F(x) \le 4$, and $17 = \sum_k \int_0^1 f_k = \int_0^1 F \le 4$, a contradiction.

Solution 2: (by Sorin Rubinstein, TAU Faculty, Tel Aviv, Israel)

Assume that each point of $(0,1)$ belongs to at most four of the intervals $I_k, 1 \le k \le n$. Let $a_1 < a_2 < \cdots < a_m$ be the endpoints of the intervals $I_k, 1 \le k \le n$, written in ascending order. Clearly, if (a_i, a_{i+1}) intersects some I_k then $(a_i, a_{i+1}) \subseteq I_k$. Let n_i be the number of intervals $I_k, 1 \le k \le n$, for which $(a_i, a_{i+1}) \subseteq I_k$. By our assumption $n_i \in \{0,1,2,3,4\}$ for every $i = 1, 2, \ldots, m-1$. Moreover the sum of the lengths of the intervals $I_k, 1 \le k \le n$, is given by $\sum_{i=1}^{m-1} n_i(a_{i+1} - a_i)$. But this leads to a contradiction since

$$\sum_{i=1}^{m-1} n_i(a_{i+1} - a_i) \le 4 \sum_{i=1}^{m-1} (a_{i+1} - a_i) = 4(a_m - a_1) \le 4 < 17$$

Hence at least one point of $(0,1)$ belongs to five or more of the intervals $I_k, 1 \le k \le n$.

The problem was also solved by:

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