

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2012 Series)

Problem:

Find an explicit number M such that if $f(x)$ is continuous on $[0, 1]$ and is twice differentiable on $(0, 1)$ and satisfies

$f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ for all x in $(0, 1)$, then $f(x) \leq M$ for all x in $[0, 1]$.

Solution 1: (by Bennett Marsh, Sophomore, ECE, Purdue University)

Let $f(x) = y$ for some $x \in (0, 1)$. Then, the slope of the secant line from $(0, f(0))$ to $(x, f(x))$ is y/x . By the mean value theorem, there exists a number a such that $0 < a < x$ and $f'(a) = y/x$. Similarly, the slope of the secant line from $(x, f(x))$ to $(1, f(1))$ is $-y/(1-x)$, and there exists a number b such that $x < b < 1$ and $f'(b) = -y/(1-x)$.

By the mean value theorem applied to $f'(x)$, there exists a number $c \in (a, b)$ such that

$$f''(c) = \frac{\frac{-y}{1-x} - \frac{y}{x}}{b-a} = \frac{-y}{x(1-x)(b-a)}.$$

But $|f''(c)| \leq 1$, so

$$\begin{aligned} \left| \frac{-y}{x(1-x)(b-a)} \right| &\leq 1 \\ \frac{|y|}{x(1-x)} &\leq b-a \leq 1 \\ |y| &\leq x(1-x) \end{aligned}$$

Over the range $[0, 1]$, the maximum value of $x(1-x)$ is $\frac{1}{4}$, so $f(x) \leq \frac{1}{4}$ for all $x \in [0, 1]$.

Solution 2: (by Pierre Castelli, Math Teacher, Antibes, France)

Let be $g(x) = \frac{1}{2}x(1-x)$ and $h(x) = f(x) - g(x)$ for all x in $[0, 1]$.

Since $h''(x) = f''(x) + 1$ and $|f''(x)| \leq 1$ for all x in $(0, 1)$, $h''(x) \geq 0$ for all x in $(0, 1)$, thus $h(x)$ is convex on $(0, 1)$.

Since $h(x)$ is also continuous on $[0, 1]$ and $h(0) = h(1) = 0$, $h(x) \leq 0$ for all x in $[0, 1]$, i.e.

$f(x) \leq g(x)$ for all x in $[0, 1]$.

As the maximum of $g(x)$ on $[0, 1]$ is $g\left(\frac{1}{2}\right) = \frac{1}{8}$, we get:

$$f(x) \leq \frac{1}{8} \quad \text{for all } x \text{ in } [0, 1].$$

Since $g(x)$ satisfies the conditions of the problem, the previous inequality is optimal.

The problem was also solved by:

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