PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2013 Series)

Problem:
Let $0 < n_1 < n_2 < \ldots$ **be integers.**
Prove

$$\sum_{i=1}^{\infty} \frac{n_{i+1} - n_i}{n_i} = \infty.$$

Solution 1: (by Carles Burnette, Graduate Student, Drexel University, PA)

Let i be an arbitrary positive integer. Since $f(x) = 1/x$ is decreasing on $(0, \infty)$, we have $1/x \leq 1/n_i$ for all $x \in [n_i, n_{i+1}]$. Therefore

$$\int_{n_i}^{n_{i+1}} \frac{1}{x} dx \leq \int_{n_i}^{n_{i+1}} \frac{1}{n_i} dx = \frac{n_{i+1} - n_i}{n_i},$$

and so for every positive integer K,

$$\sum_{i=1}^{K} \frac{n_{i+1} - n_i}{n_i} \geq \sum_{i=1}^{K} \left(\int_{n_i}^{n_{i+1}} \frac{1}{x} dx \right) = \int_{n_1}^{n_{K+1}} \frac{1}{x} dx = \log(n_{K+1}) - \log(n_1) \to \infty$$

as $K \to \infty$ since $\{n_i\}$ is a strictly increasing sequence of positive integers and is thus unbounded. It follows that $\sum_{i=1}^{\infty} \frac{n_{i+1} - n_i}{n_i} = \infty$ by direct comparison test.

Solution 2: (by Hubert Desprez, Paris, France)

Now we know that the hamonic serie $\left(H_n = \sum_{q=1}^{n} \frac{1}{q} \right)$ diverges; (with $H_0 = 0$)

$$\sum_{i=1}^{p} \frac{n_{i+1} - n_i}{n_i} = \sum_{i=1}^{p} \sum_{n=n_i}^{n_{i+1}-1} \frac{1}{n_i} \geq \sum_{i=1}^{p} \sum_{n=n_i}^{n_{i+1}-1} \frac{1}{n} = \sum_{n=n_1}^{n_{p+1}-1} \frac{1}{n} = H_{n_{p+1}-1} - H_{n_1-1} \xrightarrow{p\,\infty} \infty$$

Solution 3: (by Perfetti Paolo, Roma, Italy)

$$
\begin{aligned}
\sum_{i=p}^{q} \frac{n_{i+1}-n_i}{n_i} > \sum_{i=p}^{q} \frac{n_{i+1}-n_i}{n_{i+1}} &> \frac{1}{n_{q+1}} \sum_{i=p}^{q} (n_{i+1}-n_i) \\
&= \frac{n_{q+1}-n_{p+1}}{n_{q+1}} = 1 - \frac{n_{p+1}}{n_{q+1}} > \frac{1}{2}
\end{aligned}
$$

provided that q is large enough respect to p. This violates the Cauchy–condition and the series cannot converge. Since the terms of the series are positive, it diverges to ∞.

The problem was also solved by:

Graduates: Anuradha Bhat (Chem Engr), Tairan Yuwen (Chemistry)

Others: William Ballinger (HS student, Garfield High School, Seattle), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Shane Chern (Student, Zhejiang U. China), Hubert Desprez (Paris, France), Jon Dewitt (Student, Haverford College, PA), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Francois Seguin, Shin–ichiro Seki (Graduate Student, Osaka U, Japan), David Stigant, Lawrence R. Weill (Retired Professor, CA)