PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2013 Series)

Problem:

A standard die is rolled until a six rolls. Each time a six does not roll a fair coin is tossed, and a running tally of the number of heads minus the number of tails tossed is kept. Find the probability that the absolute value of this running tally never equals 3.
[For example, if the die rolls are 5, 2, 1, 6 and the tosses are H, H, T then the running tally is 1, 2, 1 and so in this case it never equalled 3.]

Your answer should be a fraction.

Solution 1: (by Vladimir Bar Lukianov, Afeka, Tel-Aviv, Israel)

The probability that there are n tosses of a coin is $\left(\frac{5}{6}\right)^n \cdot \frac{1}{6}$, since we need to have 1,2,3,4 or 5 for n die rolls and 6 at the $(n+1)-$ th roll.
Calculate now the probability $p(n)$ that there is no running tally of 3 or -3 during these n tosses. Observe, that for odd steps the possible values of the tally are 1 or -1, and for even steps the values are $-2, 0$ or 2. If follows, that for any $n \in \mathbb{N}$

$$p(2n) = p(2n-1)$$

since one can't get 3 or -3 just after 1 or -1. Moreover, for any $n \in \mathbb{N}$

$$p(2n+1) = \frac{3}{4}p(2n-1)$$

since there are only 2 of 8 possibilities to get 3 or -3 from 1 or -1 by two steps. So far we have for any $n \in \mathbb{N}$

$$p(2n) = p(2n-1)$$
$$p(2n+1) = \frac{3}{4}p(2n-1)$$

or in simplified manner

$$p(n) = \left(\frac{3}{4}\right)^{\lceil \frac{n-1}{2} \rceil}$$

where $\lceil \cdot \rceil$ is a sign for the nearest integer from below.

The probability that the absolute value of the running tally never equals 3 is $\dfrac{1}{6} + P$, where

$$\begin{aligned}
P =& \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n} \cdot \frac{1}{6} \cdot \left(\frac{3}{4}\right)^{\lceil \frac{n-1}{2} \rceil} \\
=& \frac{1}{6} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-1} \left(\frac{3}{4}\right)^{\lceil k-1 \rceil} + \frac{1}{6} \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k} \left(\frac{3}{4}\right)^{\lceil k-\frac{1}{2} \rceil} \\
=& \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{2k+1} \left(\frac{3}{4}\right)^{k} + \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{2k+2} \left(\frac{3}{4}\right)^{k} \\
=& \frac{1}{6} \cdot \frac{5}{6} \sum_{k=0}^{\infty} \left(\frac{25}{36} \cdot \frac{3}{4}\right)^{k} + \frac{1}{6} \cdot \frac{25}{36} \sum_{k=0}^{\infty} \left(\frac{25}{36} \cdot \frac{3}{4}\right)^{k} \\
=& \frac{1}{6} \cdot \left(\frac{5}{6} + \frac{25}{36}\right) \sum_{k=0}^{\infty} \left(\frac{25}{36} \cdot \frac{3}{4}\right)^{k} = \frac{1}{6} \cdot \frac{55}{36} \sum_{k=0}^{\infty} \left(\frac{25}{48}\right)^{k} = \frac{1}{6} \cdot \frac{55}{36} \cdot \frac{48}{23} = \frac{110}{207}.
\end{aligned}$$

Thus the answer is $\left(\dfrac{1}{6} + \dfrac{110}{207}\right) = \dfrac{289}{414}$.

Solution 2: (by Steven Landy, Physics Faculty, IUPUI)

Let $p(x) =$ the probability that the sum will reach ± 3 if it is presently x. If the sum is zero to start then there is a probability of $5/6 \cdot 1/2$ that it will be 1 after one trial, and $5/6 \cdot 1/2$ that it will be -1. By symmetry $p(1) = p(-1)$. Expanding probabilities gives

$$\begin{aligned}
p(0) &= 5/6 \cdot 1/2\, p(1) + 5/6 \cdot 1/2\, p(-1) = 5/6\, p(1) \\
p(1) &= 5/12\, p(0) + 5/12\, p(2) \\
p(2) &= 5/12\, p(1) + 5/12.
\end{aligned}$$

Solving these gives $p(0) = 125/414$. The probability that the sum never reaches 3 is $1 - p(0) = 289/414$.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Engr.)

Graduates: Anuradha Bhat (Chem Engr), Tairan Yuwen (Chemistry)

Others: Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Elie Ghosn (Montreal, Quebec), A.R. Gopinath, Levente Kornya (Portland, OR), M. Rajeswari (TA, India), Paul Richter (Jr. Niles High School), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina)