PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2013 Series)

Problem:

Let f be nonnegative, continuous, and strictly increasing on $[0, 1]$. For $p > 0$, let x_p be the number in $(0, 1)$ which satisfies

$$f^p(x_p) = \int_0^1 f^p(x)dx.$$

Find $\lim_{p\to\infty} x_p$.

Solution: (by Samson Zhou, Graduate student, CS, Purdue University)

Since f is nonnegative, continuous, and strictly increasing on $[0, 1]$, then for any $y \in (0, 1)$,

$$\int_0^1 f^p(x)dx \geq \int_y^1 f^p(x)dx$$

$$\int_0^1 f^p(x)dx \geq \int_y^1 f^p(y)dx$$

$$\int_0^1 f^p(x)dx \geq (1-y)f^p(y)$$

Hence, for any $0 < a < b < 1$, $f(a) < f(b)$, so there exists N such that for all $n > N$,

$$f^n(a) < (1-b)f^n(b).$$

That is, for any $a \in (0, 1)$, there exists N such that for all $n > N$,

$$x_n > a.$$

However, $x_p < 1$ for all p, so by the Squeeze Theorem,

$$\lim_{p\to\infty} x_p = 1.$$

The problem was also solved by:

Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), parviz Khalili (Faculty, Christopher Newport Univ.), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Dimitris Los (Athens, Greece), Sooran Mahmoudfakhe (Student, Iran), Paolo Perfetti (Roma, Italy), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina)