PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2013 Series)

Problem:
Let R be the region $\{(x,y) : 0 \le x \le 1,\ 3^x - x - 1 \le y \le x\}$. Find the volume of the solid obtained by rotating R around the line $y = x$.

Solution: (by Bennett Marsh, Junior, Physics/Math, Purdue University)

First, rotate the plane by 45° counter-clockwise so that the line $y = x$ becomes the vertical axis. The curve $y = 3^x - x - 1$ can then be described in this new coordinate system by the parametric equations

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} & t & \\ 3^t & -t & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}}t - \frac{1}{\sqrt{2}}3^t + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}3^t - \frac{1}{\sqrt{2}} \end{bmatrix}$$

The region R is now the region bounded by the $u = 0$ axis and the above curve, and the desired volume can be found by revolving the region about the v axis. This can be computed by inserting the above paremetric representations of u and v into the volume integral:

$$\begin{aligned} V = \int_0^{\sqrt{2}} \pi u^2 dv &= \pi \int_0^1 \left(\frac{2}{\sqrt{2}}t - \frac{1}{\sqrt{2}}3^t + \frac{1}{\sqrt{2}}\right)^2 \left(\frac{\log(3)}{\sqrt{2}}3^t\right) dt \\ &= \frac{\pi \log(3)}{2\sqrt{2}} \int_0^1 3^t (2t - 3^t + 1)^2 dt \\ &= \frac{\pi \log(3)}{2\sqrt{2}} \int_0^1 (4t^2 3^t - 4t 3^{2t} + 4t 3^t + 3^{3t} - 2 \cdot 3^{2t} + 3^t) dt \\ &= \frac{\pi}{\sqrt{2}} \cdot \frac{24 + (13\log(3) - 36)\log(3)}{3\log^2(3)} \\ &\approx 0.08607. \end{aligned}$$

The problem was also solved by:

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Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Kunihiko Chikaya (Kunitachi City, Japan), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Gaoyue and Gaopeng Guo (Students, Ecole Polytechnique, France), Parviz Khalili (Faculty, Christopher Newport Univ. VA), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Dimitris Los (Athens, Greece), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), David Stoner (HS Student, Aiken, S. Carolina) Bharath Swaminathan (Caterpillar, India), Aaron Tang (Student, National Univ. of Singapore), Benjamin Tsai (NIST, Gaithersburg, MD), Motohiro Tsuchiya (Graduate student, Bethesda, MD)