PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2014 Series)

Problem:

The numbers $n_1, n_2, \ldots$ are generated one at a time as follows. If $k = 1$ or $k = 2$ or $n_{k-1} \neq n_{k-2}$, roll an ordinary die once: n_k is the number rolled. If $k > 2$ and $n_{k-1} = n_{k-2}$, roll the die perhaps repeatedly until you get a number different from n_{k-1} : n_k is this different number. Prove $\lim\limits_{k\to\infty} P(n_k = n_{k+1} = 6)$ exists and find the limit.

<u>**Solution: (by Talal AL Fares, High School Teacher, Hasbaya, Lebanon)**</u>

For $k > 0$ define the event A_k : "$n_k = n_{k+1}$" and let $p_k = p(A_k)$, then

$$p(n_k = n_{k+1} = 6) = \frac{p_k}{6}.$$

We have

$$\begin{aligned} p_{k+1} &= p(A_{k+1} \cap A_k) + p(A_{k+1} \cap \overline{A_k}) \\ &= 0 + p(A_{k+1}/\overline{A_k})p(\overline{A_k}) \\ &= \left(\frac{1}{6}\right)(1 - p_k) = \frac{1 - p_k}{6}. \end{aligned}$$

Clearly $p_1 = \dfrac{1}{6}$, and by a simple induction it follows that $p_k = \dfrac{1 - \left(\frac{-1}{6}\right)^k}{7}$, which does converge to $\dfrac{1}{7}$.

Consequently, $p(n_k = n_{k+1} = 6)$ converges to $\dfrac{1}{42}$.

The problem was also solved by:

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<u>Others</u>: Hubert Desprez (Paris, France), Nathan Faber (Parker, CO), Rick Shilling & Bruce Grayson (Orlando, FL), Mohammed Hamami (AT & T), Joe Klobusicky (Geisinger Health Systems), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Jiazhen Tan (HS Student, China), Christopher J. Willy (Part-time Faculty, GWU), William Wu (Quantitative Engineering Design Inc.)