PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2014 Series)

Three Solutions by Steven Landy, IUPUI Physics staff

We look for k which satisfy both

$$\binom{n}{k+1} - \binom{n}{k} < \binom{n}{k+2} - \binom{n}{k+1}, \quad \text{and} \tag{1}$$

$$\binom{n}{k+2} - \binom{n}{k+1} \geq \binom{n}{k+3} - \binom{n}{k+2}. \tag{2}$$

Solution 1

(1) may be rewritten

$$4\binom{n}{k+1} < \binom{n}{k}+\binom{n}{k+1}+\binom{n}{k+1}+\binom{n}{k+2} = \binom{n+1}{k+1}+\binom{n+1}{k+2} = \binom{n+2}{k+2} \text{ or}$$

$$4\binom{n}{k+1} < \binom{n+2}{k+2} \tag{3}$$

When (3) is expanded we get a quadratic inequality in k which when solved gives

$$k < \frac{n-2-\sqrt{n+2}}{2} \quad \text{or} \quad k > \frac{n-2+\sqrt{n+2}}{2} \tag{4}$$

Similarly using (2) we get

$$k+1 \geq \frac{n-2-\sqrt{n+2}}{2} \quad \text{and} \quad k+1 \leq \frac{n-2+\sqrt{n+2}}{2} \tag{5}$$

Both (4) and (5) are satisfied is iff k is the greatest integer $< \dfrac{n-2-\sqrt{n+2}}{2}$. Thus there exists a unique solution to (1) and (2).

Solution 2

The problem can be restated as follows. Show that in the sequence of second order differences of $\binom{n}{k}$, the nth row of Pascal's triangle, the elements change from + to − exactly one time.

We know $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. The series of second order differences is generated by

$$(1-x)^2(1+x)^n = \sum_{k=-2}^{n} x^{k+2} \triangle^2 \binom{n}{k.} \tag{1}$$

For example for $n = 5$ we have

$$(1-x)^2(1+x)^5 = 1 + 3x + x^2 - 5x^3 - 5x^4 + x^5 + 3x^6 + x^7.$$

Let's look at row 5 of Pascal's triangle supplying zeros for non-existent coefficients. Then we subtract to get the second differences

$$\begin{array}{ccccccccccccccccccc}
0 & & 0 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & 0 & & 0 \\
 & 0 & & 1 & & 4 & & 5 & & 0 & & -5 & & -4 & & -1 & & 0 & \\
 & & 1 & & 3 & & 1 & & -5 & & -5 & & 1 & & 3 & & 1 & &
\end{array}$$

We are only asked to show that the sequence $\left\{\triangle^2 \binom{n}{k}\right\}$ where k goes from 0 to $n-2$ has a single + to − sign change. But we will show first that the entire list from $k = -2$ to $+n$ has this property. Since the lhs of (1) has no positive root except the double root at $x = 1$ there must be 2 or 4 or 6 ... sign changes on the rhs by Descarte's rule of signs. But if there are more than two sign changes on the rhs of (1), then when x is changed to $-x$ there must be less than $n+2-2$, i.e. $< n$ sign changes. But when x is changed to $-x$ there must be at least n sign changes because the lhs of (1) has a root -1 with multiplicity n (again by Descarte's rule of signs.) Thus there must be exactly two sign changes, one from + to −, and one from − to +. For $n \geq 5$ the change from + to − occurs after the $\triangle^2 \binom{n}{0}$ term. This term has value $\binom{n}{0} - 2\binom{n}{1} + \binom{n}{2} = 1 - 2n + n(n-1)/2 = (n^2 - 5n + 1)/2$, which is positive for $n = 5$ or greater.

<u>Solution 3</u>

We show that the sequence of second order differences of row n of Pascal's triangle has one change of sign from + to − and one change from − to +. Calculating the first few rows of second order differences, starting with $n = 0$, and assuming zeros for terms outside the usual range we get,

$$\begin{array}{l ccccccccccccccc}
n=0 & & & & & & 1 & & -2 & & 1 & & & & & \\
 & & & & & 1 & & -1 & & -1 & & 1 & & & & \\
 & & & & 1 & & 0 & & -2 & & 0 & & 1 & & & \\
 & & & 1 & & 1 & & -2 & & -2 & & 1 & & 1 & & \\
 & & 1 & & 2 & & -1 & & -4 & & -1 & & 2 & & 1 & \\
n=5 & 1 & & 3 & & 1 & & -5 & & -5 & & 1 & & 3 & & 1
\end{array}$$

We note that by the linearity of the operator $\triangle^2$ that the second order sequences obey the same "sum rule" as do the rows in Pascal's triangle themselves. Each element is the sum of the two elements directly above it. Since two $+$ terms sum to a $+$ and two $-$ terms to a $-$, the pattern of "$+$ block, $-$ block, $+$ block" will be preserved throughout the triangle. Note also that the sum of every row is zero (which is also inherited row after row), so that neither the inner $-$ block nor the outer $+$ blocks can ever vanish completely. The sign change from $-$ to $+$ occurs after the $k = 0$ term (as explained in solution 2) and by symmetry in the left half of the triangle. So this proves the assertion of the pow.

***Remark by the panel**: This problem gives a discrete version of the fact that normal probability density functions have exactly two inflection points.

The problem was also solved by:

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