PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2014 Series)

Problem:

If a set of real numbers is the union of a finite number of disjoint open intervals its measure is the sum of the lengths of these intervals. Find a sequence of subsets of (0,1), each the union of a finite number of disjoint open intervals, and each having measure at least $1/2$, such that the measure of the intersection of any n of these sets does not exceed $1/n$.

Solution: (by Kuang-Ru Wu, Graduate Student, Mathematics, Purdue University)

Consider $S_k = \left(0, \frac{1}{2^k}\right) \cup \left(\frac{2}{2^k}, \frac{3}{2^k}\right) \cup \cdots \cup \left(\frac{2^k-2}{2^k}, \frac{2^k-1}{2^k}\right)$, where k is a positive integer. S_k has 2^{k-1} disjoint open intervals with length $\frac{1}{2^k}$, so the measure of S_k is $\frac{1}{2}$. For any n of these sets, say $S_{l_1}, \ldots, S_{l_n}$, where the index is strictly increasing, consider the intersection of these sets. From the construction of the set, each open interval in $S_{l_{n-1}}$ contains $2^{(l_n-l_{n-1})-1}$ open intervals of S_{l_n}. Similarly, each open interval in $S_{l_{n-2}}$ contains $2^{(l_n-l_{n-2})-1}$ open intervals of $S_{l_{n-1}}$, and so on. Therefore, there are $2^{(l_n-l_{n-1})-1} \cdot 2^{(l_n-l_{n-2})-1} \cdots 2^{(l_2-l_1)-1} \cdot 2^{l_1-1}$ open intervals in the intersection, where the last term 2^{l_1-1} in the product is the number of intervals of S_{l_1}. The length of intervals in the intersection are all $\frac{1}{2^{l_n}}$, so the measure of the intersection is $\frac{1}{2^{l_n}} \cdot 2^{(l_n-l_{n-2})-1} \cdots 2^{(l_2-l_1)-1} \cdot 2^{l_1-1} = \frac{1}{2^n} \leq \frac{1}{n}$. So, the sequence S_k is what we want.

The problem was also solved by:

Undergraduates: Sameer Manchanda (Fr. CS)

Graduates: Tairan Yuwen (Chemistry)

Others: Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Jiazhen Tan (HS Student, China)