PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2014 Series)

Problem:

A standard six–sided die is rolled repeatedly and a running total is kept of all the numbers rolled. Which of 2, 6, 1006 is more likely to be one of these totals? Prove your answer.

Solution: (by Sorin Rubinstein, TAU faculty, Tel Aviv, Israel)

Let us denote for every positive integer n by p_n the probability that n is obtained as a running total. Then, $p_1 = \frac{1}{6}$ and for every $k \in \{1, 2, 3, 4, 5, 6\}$ and any positive integer $n > k$, the probability that the first number rolled is k and the running total is n equals $\frac{1}{6} \cdot p_{n-k}$.

Let us compute p_m where $m \in \{2, 3, 4, 5, 6\}$. If the running total is m the first number rolled is one of m, $m-1, m-2, \ldots, 1$. Consequently one obtains

$$p_m = \frac{1}{6} + \frac{1}{6} \cdot p_1 + \frac{1}{6} \cdot p_2 + \cdots + \frac{1}{6} \cdot p_{m-1}.$$

If follows that $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$.

For every running total $m \geq 7$ the first number rolled is one of 6, 5, 4, 3, 2, 1. Consequently one obtains

$$p_m = \frac{1}{6} \cdot p_{m-6} + \frac{1}{6} \cdot p_{m-5} + \frac{1}{6} \cdot p_{m-4} + \frac{1}{6} \cdot p_{m-3} + \frac{1}{6} \cdot p_{m-2} + \frac{1}{6} \cdot p_{m-1}$$

which is the arithmetic mean value of $p_{m-6}, p_{m-5} \ldots, p_{m-1}$. Since the mean value is smaller than or equal to the biggest of the involved numbers with equality only in the case that all the involved numbers are equal it follows that $p_m < p_6$ for $m = 7, 8, \ldots, 12$ and, subsequently for every $m \geq 7$.

Hence, it is more likely to obtain 6 than any other running total.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Physics & Math), Bryan Rainey (Jr. CS)

Graduates: Tairan Yuwen (Chemistry)

Others: Marco Biagini (Math Teacher, Italy), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL),

Ghasem Esmati (Sharif Univ. of Tech), Mohammed Hamami (AT & T), Kipp Johnson (Valley Catholic HS teacher, Oregon), Sachin Kalia (Graduate Student, U of Minnesota), Joe Klobusicky (Geisinger Health Systems), Peter Kornya (Retired Faculty, Ivy Tech), Jacob H. Bechtel & Allison Kunz (Students, Christopher Newport U), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Esmaeil Parsa (Lecturer, Iran), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Mark Senn (Systems Programmer, Purdue Univ.), David Stoner (HS Student, Aiken, S. Carolina), Michael Tomaine (Bellevue, WA), William Wu (Quantitative Engineering Design Inc.), Xu Zhang (Visiting Assistant Professor, Math, Purdue)