PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2014 Series)

Problem:

Start with a five–tuple of numbers (a, b, c, d, e) and call numbers which are next to each other neighbors. Also the first and last entries are neighbors. Make a new five–tuple by replacing each entry by the average of its neighbors: $\left(\frac{(e+b)}{2}, \frac{(a+c)}{2}, \frac{(b+d)}{2}, \frac{(c+e)}{2}, \frac{(d+a)}{2}\right)$. Next make a third five–tuple from the second in the same manner, and iterate this process indefinitely. Prove that as the number of iterations approaches infinity the five–tuples approach (q, q, q, q, q) for some number q.

Solution 1: (by Gruian Cornel, Cluj-Napoca, Romania)

[Remark: This is exactly Gruian Cornel's solution, to whom we apologize for our original (and incorrect) version of his solution.]

In the space $\mathbb{R}^5$ let us define $v_0 = (q, q, q, q, q)$ where $q = (a+b+c+d+e)/5$, $v_1 = (a, b, c, d, e)$ and for any $n \geq 1$, $v_{n+1} = f(v_n)$ where $f : \mathbb{R}^5 \to \mathbb{R}^5$ is the function who realize the

iteration. Now observe that $v_4 = \frac{1}{2}v_2 - \frac{1}{8}v_1 + \frac{5}{8}v_0$. Obviously f is linear, therefore for any $n \geq 2$, $v_{n+2} = \frac{1}{2}v_n - \frac{1}{8}v_{n-1} + \frac{5}{8}v_0$, so $v_{n+2} - v_0 = \frac{1}{2}(v_n - v_0) - \frac{1}{8}(v_{n-1} - v_0)$ and so

$$\|v_{n+2} - v_0\| \leq \frac{1}{2}\|v_n - v_0\| + \frac{1}{8}\|v_{n-1} - v_0\| \leq \frac{5}{8}\|v_m - v_0\|, m \in \{n-1, n\},$$
$$\|v_m - v_0\| = \max\{\|v_n - v_0\|, \|v_{n-1} - v_0\|\}.$$

Therefore $\|v_{n+2} - v_0\| \leq \left(\frac{5}{8}\right)^p \|v_r - v_0\|$ where $r \in \{1, 2, 3\}$ and $p \geq \left\lfloor \frac{n}{3} \right\rfloor - 1$. Hence $v_n \overset{n\to\infty}{\longrightarrow} v_0$.

Solution 2: (by Sorin Rubinstein, TAU faculty, Tel Aviv, Israel)

We identify the five-tuple (a, b, c, d, e) with the vector $v = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$. Let us define the matrix

$T = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$. We must show that for some q $\lim_{n\to\infty} T^n v = \begin{pmatrix} q \\ q \\ q \\ q \\ q \end{pmatrix}$. By a straightforward (but lengthy) calculation one verifies that the characteristic polynomial of T is $(\lambda^2 + \frac{1}{2}\lambda - \frac{1}{4})^2(\lambda - 1)$. Hence the eigenvalues of T are $\lambda_1 = \lambda_2 = \frac{-1+\sqrt{5}}{4}$, $\lambda_3 = \lambda_4 = \frac{-1-\sqrt{5}}{4}$ and $\lambda_5 = 1$. We remark that $|\lambda_j| < 1$ for $j = 1, 2, 3, 4$.

Since T is symmetric there exists an orthogonal matrix U such that

$T = U \cdot \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot U^*$. Consequently:

$T^n = U \cdot \begin{pmatrix} \lambda_1^n & 0 & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 & 0 \\ 0 & 0 & \lambda_3^n & 0 & 0 \\ 0 & 0 & 0 & \lambda_4^n & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot U^*$. On the other hand since clearly

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of T associated to the eigenvalue $\lambda_5 = 1$ the last column of U is

$\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$. Therefore

$$\lim_{n\to\infty} T^n v = U \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot U^* v = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \cdot \left[\frac{1}{\sqrt{5}} \; \frac{1}{\sqrt{5}} \; \frac{1}{\sqrt{5}} \; \frac{1}{\sqrt{5}} \; \frac{1}{\sqrt{5}} \right] \cdot \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} q \\ q \\ q \\ q \\ q \end{pmatrix}$$

where $q = \frac{a+b+c+d+e}{5}$.

Solution 3: (by Yang Mo, Sophomore, Physics, Purdue University)

Idea: Make a recurrence relation based on the process and solve the difference equation. Convince yourself that before and after the transform, the sum of five number is the same:

$$a+b+c+d+e = \frac{e+b}{2} + \frac{a+c}{2} + \frac{b+d}{2} + \frac{c+e}{2} + \frac{d+a}{2}.$$

Let $q = \dfrac{a+b+c+d+e}{5}$. Make the number in the first place a_n, the second place b_n and so on. We have from problem $a_{n+1} = \dfrac{e_n+b_n}{2}, b_{n+1} = \dfrac{a_n+c_n}{2}, e_{n+1} = \dfrac{d_n+a_n}{2} \ldots$.
Then $a_{n+2} = \dfrac{e_{n+1}+b_{n+1}}{2} = \dfrac{\frac{a_n+c_n}{2}+\frac{d_n+a_n}{2}}{2} = \dfrac{2a_n+c_n+d_n}{4} = \dfrac{a_n+5q-(e_n+b_n)}{4} = \dfrac{5q+a_n-2a_{n+1}}{4}$.
Rewriting the equation we get $4a_{n+2} = 5q + a_n - 2a_{n+1}$. Let $a'_{n+2} = a_{n+2} - q$ same for a'_n and a'_{n+1}.
$4a'_{n+2} + 2a'_{n+1} - a'_n = 0$. Notice this is a second order linear difference equation (homogenous and with constant coefficients). The corresponding characteristic equation is (similar to solution to second order ODE).
$4x^2 + 2x - 1 = 0$. The solutions are $\dfrac{-1+\sqrt{5}}{4}$ and $\dfrac{-1-\sqrt{5}}{4}$. Then a'_n should be in this form: $a'_n = C_1\left(\dfrac{-1+\sqrt{5}}{4}\right)^n + C_2\left(\dfrac{-1-\sqrt{5}}{4}\right)^n$, where Cs are constants.
Without solving the equation, we see as n approaches infinity both terms vanish. At the point of infinity, $a'_n = 0$ and $a_n = q = \dfrac{a+b+c+d+e}{5}$.
The reasoning works as well for the other four coordinates.

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[Note from the panel. Specialized knowledge about Markov and Toeplitz matrices were used by some solvers to simplify matrix calculations. Another method of solution used by several solvers was to note that the maximum of the five numbers after the n^{th} iteration is nonincreasing in n and the minimum is non decreasing, and then to show that the limits of the maximum and minimum are the same.] ***

The problem was also solved by:

Undergraduates: Bryan Rainey (Jr. CS)

Graduates: Kuang-Ru Wu (Math), Tairan Yuwen (Chemistry)

Others: Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Rick Shilling & Bruce Grayson (Orlando, FL), Kipp Johnson (Valley Catholic HS teacher, Oregon), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Vladimir B. Lukianov (Lecturer, Tel-Aviv), Perfetti Paolo (Roma, Italy), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Mehtaab Sawhney (HS Student, Commack HS, NY), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), David Stoner (HS Student, Aiken, S. Carolina), Jiazhen Tan (HS Student, China), Aaron Tang (Student, National Univ. of Singapore), William Wu (Quantitative Engineering Design Inc.)