## PROBLEM OF THE WEEK Solution of Problem No. 6 (Fall 2014 Series)

**Problem:** 

Prove that if  $a_1, a_2, \ldots$  is a sequence of nonnegative numbers and if  $a_n \leq \frac{(a_{n-1}+a_{n-2})}{n}$ , for n > 2, then  $\sum_{n=1}^{\infty} a_n$  converges.

## Solution 1: (by Talal AL Fares, Math Instructor, Hasbaya, Lebanon)

Let  $M = \max\{a_1, a_2\}$ , and let's prove by induction that

(1) 
$$a_n \le \frac{M}{2^{n-3}}$$
 for all  $n \ge 1$ .

For n = 1, 2 clear.  $a_3 \leq \frac{a_1 + a_2}{3} \leq \frac{2M}{3} \leq M$ . Similarly,  $a_4 \leq \frac{a_3 + a_2}{4} \leq \frac{M + \frac{2M}{3}}{4} \leq \frac{M}{2}$ . Now, let n > 4 and suppose  $a_i \leq \frac{M}{2^{i-3}}$  for all  $i \leq n$ , then

$$(n+1)a_{n+1} \le a_n + a_{n-1} \le \frac{M}{2^{n-3}} + \frac{M}{2^{n-4}} = \frac{3M}{2^{n-3}}$$

and consequently  $a_{n+1} \leq \frac{M}{2^{n-2}}$  since  $n+1 \geq 6$ . Thus (1) is true. But  $\sum_{n>0} 2^{3-n} = 8$  so  $\sum_{n>0} a_n$  converges (as  $a_n \geq 0 \ \forall n$ ).

## Solution 2: (by Paolo Perfetti, Roma, Italy)

Let  $S \doteq \sum_{n=1}^{\infty} a_n$  and  $S_N = \sum_{n=1}^{N} a_n$ . The nonnegativity of  $a_n$  makes  $\sum_{n=1}^{\infty} a_n$  either finite or  $+\infty$ . If n > 3,  $3a_n < na_n < a_{n-1} + a_{n-2}$ 

whence

$$3\sum_{n=3}^{N} a_n \le \sum_{n=3}^{N} a_{n-1} + \sum_{n=3}^{N} a_{n-2},$$

 $\mathbf{SO}$ 

$$3(S_N - a_1 - a_2) \le S_N - a_1 + S_N \iff S_N \le 2a_1 + 3a_2$$

and this is impossible if  $S = +\infty$ . The only possible conclusion is that S is finite.

## The problem was also solved by:

Undergraduates: Scott Kelly (Jr. CS), Yang Mo (So. Phys), Bryan Rainey (Jr. CS)

Graduates: Kuang-Ru Wu (Math), Tairan Yuwen (Chemistry)

<u>Others</u>: Vanio Beccaccioli (Terni, Italy), KD Harald Bensom (Germany), Marco Biagini (Math Teacher, Italy), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Kipp Johnson (Valley Catholic HS teacher, Oregon), Sachin Kalia (Graduate Student, U of Minnesota), Joe Klobusicky (Geisinger Health Systems), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Matthew Lim, Vladimir B. Lukianov (Lecturer, Tel-Aviv), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Mehtaab Sawhney (HS Student, Commack HS, NY), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Leo Sheck (Auckland, NZ), Jiazhen Tan (HS Student, China), Aaron Tang (Student, National Univ. of Singapore), Xu Zhang (Visiting Assistant Professor, Math, Purdue)