# PROBLEM OF THE WEEK 

Solution of Problem No. 6 (Fall 2014 Series)

## Problem:

Prove that if $a_{1}, a_{2}, \ldots$ is a sequence of nonnegative numbers and if $a_{n} \leq$ $\frac{\left(a_{n-1}+a_{n-2}\right)}{n}$, for $n>2$, then $\sum_{n=1}^{\infty} a_{n}$ converges.

## Solution 1: (by Talal AL Fares, Math Instructor, Hasbaya, Lebanon)

Let $M=\max \left\{a_{1}, a_{2}\right\}$, and let's prove by induction that

$$
\text { (1) } \quad a_{n} \leq \frac{M}{2^{n-3}} \quad \text { for all } n \geq 1
$$

For $n=1,2$ clear. $a_{3} \leq \frac{a_{1}+a_{2}}{3} \leq \frac{2 M}{3} \leq M$. Similarly, $a_{4} \leq \frac{a_{3}+a_{2}}{4} \leq \frac{M+\frac{2 M}{3}}{4} \leq \frac{M}{2}$. Now, let $n>4$ and suppose $a_{i} \leq \frac{M}{2^{i-3}}$ for all $i \leq n$, then

$$
(n+1) a_{n+1} \leq a_{n}+a_{n-1} \leq \frac{M}{2^{n-3}}+\frac{M}{2^{n-4}}=\frac{3 M}{2^{n-3}}
$$

and consequently $a_{n+1} \leq \frac{M}{2^{n-2}}$ since $n+1 \geq 6$. Thus (1) is true.
But $\sum_{n>0} 2^{3-n}=8$ so $\sum_{n>0} a_{n}$ converges (as $a_{n} \geq 0 \forall n$ ).

## Solution 2: (by Paolo Perfetti, Roma, Italy)

Let $S \doteq \sum_{n=1}^{\infty} a_{n}$ and $S_{N}=\sum_{n=1}^{N} a_{n}$. The nonnegativity of $a_{n}$ makes $\sum_{n=1}^{\infty} a_{n}$ either finite or $+\infty$.

$$
\text { If } n \geq 3, \quad 3 a_{n} \leq n a_{n} \leq a_{n-1}+a_{n-2}
$$

whence

$$
3 \sum_{n=3}^{N} a_{n} \leq \sum_{n=3}^{N} a_{n-1}+\sum_{n=3}^{N} a_{n-2},
$$

so

$$
3\left(S_{N}-a_{1}-a_{2}\right) \leq S_{N}-a_{1}+S_{N} \Longleftrightarrow S_{N} \leq 2 a_{1}+3 a_{2}
$$

and this is impossible if $S=+\infty$. The only possible conclusion is that $S$ is finite.

## The problem was also solved by:

Undergraduates: Scott Kelly (Jr. CS), Yang Mo (So. Phys), Bryan Rainey (Jr. CS)

Graduates: Kuang-Ru Wu (Math), Tairan Yuwen (Chemistry)
Others: Vanio Beccaccioli (Terni, Italy), KD Harald Bensom (Germany), Marco Biagini (Math Teacher, Italy), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Kipp Johnson (Valley Catholic HS teacher, Oregon), Sachin Kalia (Graduate Student, U of Minnesota), Joe Klobusicky (Geisinger Health Systems), Steven Landy (Physics Faculty, IUPUI), WeiXiang Lien (Miaoli, Taiwan), Matthew Lim, Vladimir B. Lukianov (Lecturer, Tel-Aviv), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Mehtaab Sawhney (HS Student, Commack HS, NY), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Leo Sheck (Auckland, NZ), Jiazhen Tan (HS Student, China), Aaron Tang (Student, National Univ. of Singapore), Xu Zhang (Visiting Assistant Professor, Math, Purdue)

