PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2001 Series)

Problem: Given a triangle ABC, choose A_1, B_1, C_1 on the sides opposite A, B, C respectively so that the centroid of $A_1B_1C_1$ coincides with that of ABC. Determine (with proof) the locations of A_1, B_1, C_1 so that the ratio of the area of $A_1B_1C_1$ to that of ABC is minimal.

Solution (by the Panel)

An affine transformation of the plane leaves centroids and ratios of corresponding areas invariant. Make an affine transformation that puts the vertices A, B, C at $(0,1), (0,0), (1,0)$ of the xy-plane. Then the coordinates of C_1, A_1, B_1 are $(0,r), (s,0), (t,1-t)$ resp., where $r, s, t \in (0,1)$. The centroid of $\triangle ABC$ is at $(\frac{1}{3}, \frac{1}{3})$, that of $\triangle A_1B_1C_1$ is at $(\frac{1}{3}(s+t), \frac{1}{3}(r+1-t))$, hence $s+t=1$, $r+1-t=1$, so $r=t$, $s=1-t$. Then $2\ \mathrm{area}(\triangle ABC) = 1$,

$$2\ \mathrm{area}(\triangle A_1B_1C_1) = \begin{vmatrix} 0 & t & 1 \\ 1-t & 0 & 1 \\ t & 1-t & 1 \end{vmatrix} = 3t^2 - 3t + 1 = 3(t-\frac{1}{2})^2 + \frac{1}{4} \geq \frac{1}{4},$$

and equality holds if and only if $t = \frac{1}{2}$. So minimal ratio is $\frac{1}{4}$ and it is attained if and only if A_1, B_1, C_1 are the midpoints of the sides of $\triangle ABC$.

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Two unacceptable solutions of Problem 1 were received late.