PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2001 Series)

Problem: Suppose f is a polynomial in n variables, of degree $\leq n-1$ $(n = 2, 3, \cdots)$. Prove the identity $\sum(-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_n} f(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = 0$, where ϵ_i is either 1 or 0 and the sum is over all of the 2^n combinations.

Solution (by Steven Landy, Fac. Physics at IUPUI, edited by the Panel)

The identity is linear in f, so it suffices to prove it for f of the form $f(x_1, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $p_1 + p_2 + \cdots + p_n \leq n-1$. Because of this last restriction, at least one of the p_i is 0, say $p_n = 0$. Then writing the whole sum as the sum of the terms with $\epsilon_n = 0$ and those with $\epsilon_n = 1$, we have

$$S = (-1)^0 \sum_{\epsilon_1,\ldots,\epsilon_{n-1}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \epsilon_1^{p_1} \cdots \epsilon_{n-1}^{p_{n-1}} + (-1)^1 \sum_{\epsilon_1,\ldots,\epsilon_{n-1}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \epsilon_1^{p_1} \cdots \epsilon_{n-1}^{p_{n-1}},$$

which is the difference of two identical terms, hence $S = 0$.

Also solved by:

<u>Undergraduates</u>: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. MA)

<u>Graduates</u>: Amit Shirsat (CS), Thierry Zell (MA)

<u>Others</u>: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Three unsatisfactory solutions were received.

Two correct late solutions were received that were mailed from abroad before their deadlines:

Prob. 1 Martin Lukarvoski (Undergrad., Skopje, Macedonia)

Prob. 3 Julien Santini (High School, Paris, France)