PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2001 Series)

Problem: An integer n has property P if there are integers p and q such that $0 < p < q < n$ and the sum $p + (p+1) + \cdots + q$ is dvisible by n. Show that n has property P if and only if n is not a power of 2.

Solution (by Mike Hamburg, Jr. at St. Joseph HS, South Bend, IN, edited by the Panel)

(i) Suppose n is not a power of 2, $n = 2^r(2k+1)$, $r \geq 0, k \geq 1$. Let $a = \max(2^{r+1}, 2k+1)$, $b = \min(2^{r+1}, 2k+1)$. Then $n \geq a > b \geq 2$; also a and b are of opposite parity. Now let

$$p = \frac{a-b+1}{2}\ , \quad q = \frac{a+b-1}{2}.$$

Both p and q are integers and $0 < p < q < n$. Also

$$p + (p+1) + \cdots + q = \frac{1}{2}(p+q)(q-p+1) = \frac{1}{2}ab = n,$$

so n divides the sum (is actually equal to the sum).

(ii) Suppose $n = 2^k (k > 1)$ and $n|\frac{1}{2}(p+q)(q-p+1)$, then $2^{k+1}|(p+q)(q-p+1)$. One of the two factors is odd, so 2^{k+1} divides either $(p+q)$ or $(q-p+1)$. But
$2^{k+1} = 2n > p+q > q-p+1$. This is a contraction.

Also complete or partially solved by:

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<u>Graduates</u>: Gajath Gunatillake (MA), Ashish Rao (ECE), Amit Shirsat (CS)

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Three unacceptable solutions were received.