PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2001 Series)

Problem: Evaluate $I = \iint_R \frac{1}{(x^2+1)y}\,dxdy$ where R is the region in the upper half plane between the two curves $2x^4 + y^4 + y = 2$, $x^4 + 8y^4 + y = 1$.

Solution (by Mike Hamburg, St. Joseph's H.S.)

Let $f(x) = y \geq 0$ such that $x^4+8y^4+y = 1$. There is only one $y \geq 0$ satisfying this because $8y^4+y$ is monotone increasing for $y \geq 0$. Then $2x^4+16y^4+2y = 2$, so $2x^4+(2y)^4+(2y) = 2$, so $2f(x)$ is the upper limit. Also note that $f(x)$ is defined only for $|x| \leq 1$. Then

$$\iint_R \frac{dxdy}{(x^2+1)y} = \int_{-1}^{1} \left(\int_{f(x)}^{2f(x)} \frac{dy}{y} \right) \frac{dx}{1+x^2} = \int_{-1}^{1} \left(\log y \Big|_{f(x)}^{2f(x)} \right) \frac{dx}{1+x^2}$$
$$= \int_{-1}^{1} (\log(2f(x)) - \log(f(x))) \frac{dx}{1+x^2}.$$

But

$$\log 2f(x) - \log f(x) = \log \frac{2f(x)}{f(x)} = \log 2$$

for all x, so this becomes

$$\int_{-1}^{1} \frac{\log 2}{1+x^2}\,dx = (\log 2)\tan^{-1} x \Big|_{-1}^{1} = (\log 2)(\frac{\pi}{4} - (-\frac{\pi}{4})) = \frac{\pi \log 2}{2}.$$

Also solved by:

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