PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2001 Series)

Problem: Suppose A, B are real $n \times n$ matrices with $A+B=I$ (identity) and $\operatorname{rank}(A)+\operatorname{rank}(B)=n$. Show that $A^2=A$, $B^2=B$, $AB=BA=0$.

Solution (by Vikram Buddhi, Grad. MA, edited by the Panel)

Let V be a vector space of dimension n, on which A and B act. Let $\operatorname{rank}(A)=r$, so $\operatorname{nullity}(A)=n-r$, $\operatorname{nullity}(B)=r$. Let $x \in \operatorname{kernel}(A) \cap \operatorname{kernel}(B)$. Then $Ax=0$ and $(I-A)x=0$, $\therefore x=0$. Hence $V=\ker(A) \oplus \ker(B)$. Let arbitrary $x \in V$ be decomposed: $x=x_1+x_2$, $x_1 \in \ker(A)$, $x_2 \in \ker(B)$. Then $BAX = BAX_1 + BAX_2 = 0 + ABX_2 = 0$, $\therefore BA=0$, likewise $AB=0$. Also $A-A^2=AB=0$, $A=A^2$; $B-B^2=BA=0$, $B=B^2$.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA), Yee-Ching Yeow (Jr. Math)

<u>Graduates</u>: Dharmashankar Subramanian (CHE)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Two incorrect solutions were received.