PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2001 Series)

Problem: Let p be a prime number and let J be the set of all 2×2 matrices, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \{0, 1, \cdots, p-1\}$, and which satisfy $a + b \equiv 1 \pmod{p}$ and $ad - bc \equiv 0 \pmod{p}$. How many matrices are in J?

Solution (by Steven Landy, Fac. Phys. at IUPUI)

$\underline{a}$ can take on p values: $0, 1, \ldots, p-1$; $b \equiv 1 - a$ is then fixed.

If $a \equiv 0$ then $b \equiv 1, c \equiv 0$, while d can be one of $0, 1, \ldots, p-1$.

If $a \equiv 1$ then $b \equiv 0, d \equiv 0$, while c can be one of $0, 1, \ldots, p-1$.

If $a \not\equiv 0, a \not\equiv 1$, then $b \not\equiv 0$ and in $ad \equiv bc$, d can be any of $0, 1, \ldots, p-1$; and $c \equiv adb^{-1}$, where b^{-1} is the unique reciprocal of $b \not\equiv 0 \pmod{p}$.

Thus, for any choice of $\underline{a}$ there are p ways to assign the remaining terms. Hence, the cardinality of J is p^2.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA)

<u>Others</u>: Jonathan Landy (Jr. Warren Central H.S., Indpls.), Julien Santini (Lacordaire H.S., France), Aditya S. Utturwar (Aero, Georgia Tech)