PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2002 Series)

Problem: Define the sequence $\{a_n\}, n = 0, 1, 2, \cdots$ as follows: $a_0 = 1+i,\ a_n = a_{n-1}^{1+i}$ $(n = 1, 2, \cdots)$. Determine the real part of a_{8k+1} for $k = 0, 1, 2, \ldots$.

Solution (by the Panel)

We meant the definition of a_n to be

$$a_n = a_0^{(1+i)^n}. \tag{1}$$

With this interpretation the problem has the following solution:

$$(1+i)^8 = 2^4,\ (1+i)^{8k+1} = 2^{4k}(1+i)$$

$$a_{8k+1} = (1+i)^{2^{4k}(1+i)} = (e^{\frac{1}{2}\ell n2 + i\pi/4})^{2^{4k}(1+i)} = 2^{2^{4k-1}} e^{-2^{4k-2}\pi} e^{2^{4k-1}(\ell n2)i} e^{2^{4k-2}\pi i}$$

so

$$\text{Re } a_{8k+1} = 2^{2^{4k-1}} e^{-2^{4k-2}\pi} \cos(2^{4k-1}\ell n2) \quad \text{for } k \geq 1,$$

since $e^{2^{4k-2}\pi i} = 1$ if $k \geq 1$.

However, the problem did not define a_n as in (1), but

$$a_n = a_{n-1}^{1+i}, \tag{2}$$

e.g. $a_2 = (a_0^{1+i})^{1+i}$, which is not the same as $a_2 = a_0^{(1+i)^2} = a_0^{2i}$. If u, v are real, $(a^u)^v = a^{uv}$, but this is not true if u, v are complex, e.g. $(e^{2\pi i})^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$, but $e^{(2\pi i)(\frac{1}{2})} = e^{\pi i} = -1$. So we have $a_2 = a_1^{1+i} = [2^{\frac{1}{2}} e^{-\pi/4} e^{(\pi/4 + \frac{1}{2}\ell n2)i}]^{1+i}$, which is already so complicated that it is hopeless to calculate a_{8k+1}. We are giving credit to those who got the above solution. We apologize to those who may have used the stated version (2), spent time and effort and gave up, not handing in anything. There are a couple of solvers who used still another version:

$$a_{8k+1} = a_{8k}^{1+i} = (2^{2^{4k-1}})^{1+i} \tag{3}$$

which gives a much simpler result than the above one. We also gave full credit to these solvers.

Solved by:

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Two unacceptable solutions were received.