PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2002 Series)

Problem: Determine the number a for which $\int_0^\pi [\sin x - ax(\pi - x)]^2 dx$ is minimal.

Solution (by the Panel)

We present our own solution, which avoids the testing of the critical point of the quadratic polynomial

$$I(a) = a^2 \int_0^\pi x^2(\pi - x)^2 dx - 2a \int_0^\pi x(\pi - x)\sin x\, dx + \int_0^\pi \sin^2 x\, dx.$$

Carrying out the integrations (the main tool is integration by parts), one obtains

$$\begin{aligned} I(a) &= \frac{\pi^5}{30}\, a^2 - 8a + \frac{\pi}{2} \\ &= \frac{\pi^5}{30}\, (a - \frac{120}{\pi^5})^2 + \frac{\pi}{2} - \frac{480}{x^5} \geq \frac{\pi}{2} - \frac{480}{\pi^5}, \end{aligned}$$

and equality holds in the last inequality if and only if $a = 120/\pi^5$. Therefore, this is the value for which $I(a)$ is a minimum.

Solved by:

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Three incorrect solutions were received.