PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2002 Series)

Problem: For $e \neq 0$, determine the roots of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ as functions of a, d, and e, given that the equation has two roots whose product is 1 and two other roots whose product is -1.

Solution (by the Panel)

Let the roots be $p, \frac{1}{p}, q, -\frac{1}{q}, r$.

(1) $p \cdot \frac{1}{p} \cdot q \cdot \frac{-1}{q} \cdot r = -e \quad \text{so} \quad r = e.$

(2) $(p + \frac{1}{p}) + (q - \frac{1}{q}) + r = -a \quad \text{so} \quad (p + \frac{1}{p}) + (q - \frac{1}{q}) = -(a + e).$

(3) $\frac{-e}{p} - ep - \frac{e}{q} + eq - \frac{e}{r} = d \quad \text{so} \quad (p + \frac{1}{p}) - (q - \frac{1}{q}) = -\frac{d+1}{e}.$

From this

$$p + \frac{1}{p} = \frac{1}{2}(-(a + e)) - \frac{d + 1}{e}) = -\frac{1}{2e}(e^2 + ae + d + 1) = A,$$
$$q - \frac{1}{q} = \frac{1}{2}(\frac{d + 1}{e} - (a + e)) = -\frac{1}{2e}(e^2 + ae - d - 1) = B.$$

Then $p^2 - Ap + 1 = 0$ so $p = \frac{1}{2}(A \pm \sqrt{A^2 - 4})$ and $\frac{1}{p} = A - p = \frac{1}{2}(A \mp \sqrt{A^2 - 4})$.

We may use the plus sign for p and the minus sign for $\frac{1}{p}$. Similarly
$q = \frac{1}{2}(B + \sqrt{B^2 + 4})$ $\frac{1}{q} = \frac{1}{2}(B - \sqrt{B^2 + 4})$.

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