

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2002 Series)

Problem: Sum the series $\frac{1}{4!} + \frac{4!}{8!} + \frac{8!}{12!} + \frac{12!}{16!} + \dots$

Solution (by Chit Hong Yam (Fr. Engr.))

$$\begin{aligned} S &= \frac{1}{4!} + \frac{4!}{8!} + \frac{8!}{12!} + \frac{12!}{16!} + \dots = \sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{6(4k+1)} - \frac{1}{2(4k+2)} + \frac{1}{2(4k+3)} - \frac{1}{6(4k+4)} \right]. \end{aligned}$$

Now consider the following definite integrals:

$$\int_0^1 x^{4k} dx = \frac{1}{4k+1}; \int_0^1 x^{4k+1} dx = \frac{1}{4k+2}; \int_0^1 x^{4k+2} dx = \frac{1}{4k+3}; \int_0^1 x^{4k+3} dx = \frac{1}{4k+4}.$$

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \int_0^1 \left[\frac{1}{6}x^{4k} - \frac{1}{2}x^{4k+1} + \frac{1}{2}x^{4k+2} - \frac{1}{6}x^{4k+3} \right] dx \\ &= \int_0^1 \sum_{k=0}^{\infty} \left[\frac{1}{6}x^{4k} - \frac{1}{2}x^{4k+1} + \frac{1}{2}x^{4k+2} - \frac{1}{6}x^{4k+3} \right] dx \\ &= \frac{1}{6} \int_0^1 \sum_{k=0}^{\infty} x^{4k} (1 - 3x + 3x^2 - x^3) dx \\ &= \frac{1}{6} \int_0^1 \frac{1}{1-x^4} (1-x)^3 dx = \frac{1}{6} \int_0^1 \frac{(1-x)^2}{(1+x^2)(1+x)} dx = \frac{1}{6} \int_0^1 \left[\frac{2}{1+x} - \frac{x+1}{1+x^2} \right] dx. \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \left[\frac{1}{3} \ln(1+x) - \frac{1}{12} \ln(1+x^2) - \frac{1}{6} \tan^{-1} x \right]_0^1 \\ &= \left(\frac{1}{3} - \frac{1}{12} \right) \ln 2 - \frac{1}{6} \tan^{-1}(1). \end{aligned}$$

$$\text{Evaluating, } S = \frac{1}{4} \ln 2 - \frac{\pi}{24}.$$

Also solved by:

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Three incorrect solutions were received.