PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2002 Series)

Problem: Suppose a, b are relatively prime odd positive integers. Show that $\sum_{0<m<\frac{b}{2}}[\frac{a}{b}m]+\sum_{0<n<\frac{a}{2}}[\frac{b}{a}n] = \frac{a-1}{2}\cdot\frac{b-1}{2}$, where $[x]$ denotes the largest integer $\leq x$.

Solution (by J.L.C. in Fishers, IN; edited by the Panel)

In a Cartesian coordinate system, consider $R = \{(m,n)|m,n$ are integers, and $1 \leq m \leq \frac{b-1}{2},\ 1 \leq n \leq \frac{a-1}{2}\}$ and also consider straight line $\ell : y = \frac{a}{b}x$, which separates R into two parts: one part has points right-below ℓ, and the other part has points left-above ℓ. Since $[\frac{a}{b}x] < \frac{a}{b}x$, for given $m = 1, 2, \cdots, \frac{b-1}{2}$, $[\frac{a}{b}m]$ is the number of points in R located right-below ℓ, therefore $\sum_{m=1}^{\frac{b-1}{2}}[\frac{a}{b}m]$ is the total number of points in R located right-below ℓ. From $y = \frac{a}{b}x$, $x = \frac{b}{a}y$, and $[\frac{b}{a}y] < \frac{b}{a}g$, for given $n = 1, 2, \cdots, \frac{n-1}{2}$, $[\frac{b}{a}n]$ is the number of points in R located left-above ℓ. Therefore $\sum_{n=1}^{\frac{a-1}{2}}[\frac{b}{a}n]$ is the total number of points in R located left-above ℓ. But R has exactly $\frac{a-1}{2}\cdot\frac{b-1}{2}$ points, hence

$$\sum_{m=1}^{\frac{b-1}{2}}[\frac{a}{b}m] + \sum_{n=1}^{\frac{a-1}{2}}[\frac{b}{a}n] = \frac{a-1}{2}\cdot\frac{b-1}{2}.$$

Also solved by:

<u>Undergraduates</u>: Yue Wei Lu (So. EE/MA),

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Leo Sheck (Medical Sch, U. Auckland, NZ), Aditya Utturwar (Grad. AE, Georgia Tech)