PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2002 Series)

Problem: Determine all the real 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $A^2 + A + I = 0$.

Solution (by Damir Dzhafarov, Fr. MA, edited by the Panel)

Expanding the equation yields the system

$$\begin{cases} a^2 + a + bc + 1 = 0 & \text{(i)} \\ ab + bd + b = 0 & \text{(ii)} \\ ac + cd + c = 0 & \text{(iii)} \\ d^2 + d + bc + 1 = 0 & \text{(iv)} \end{cases}$$

Solving (i) and (iv) for a and d respectively gives $a = \frac{-1 \pm \sqrt{-3-4bc}}{2}$ and $d = \frac{-1 \pm \sqrt{-3-4bc}}{2}$, whence it follows that $bc \leq -3/4$ and consequently $b \neq 0$ and $c \neq 0$. Thus, (ii) may be divided by b and (iii) by c to obtain $a + d = -1$, which is satisfied only if $bc \leq -3/4$. The sought solutions are therefore

$$A = \begin{pmatrix} \frac{-1 \pm \sqrt{-3-4mn}}{2} & m \\ n & \frac{-1 \mp \sqrt{-3-4mn}}{2} \end{pmatrix},$$

where m, n arbitrary real numbers except $mn < -3/4$.

Also solved by:

Undergraduates: Eric Tkaczyk (Jr. EE/MA)

Graduates: Dharmashankar Subramanian (ChE)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Dane Brooke (Boeing, Seattle), J.L.C. (Fishers, IN), John G. DelGreco (MA, Loyola U.), Leo Sheck (Medical Sch, U. Auckland, NZ),

Five unacceptable solutions were received.