PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2002 Series)

Problem: If P, Q, R, S are polynomials, show that $\int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$ is divisible by $(x-1)^4$.

Solution (by Chris Lomont, graduate (MA), edited by the Panel)

Let $F(x) = \int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$. Clearly $F(x)$ is a polynomial in x. $F(1) = 0$ (clearly: $\int_1^1 \cdots = 0$), so $(x-1)$ divides $F(x)$.

$$\begin{aligned} F'(x) = (RS)(x) \int_1^x PQ + (PQ)(x) \int_1^x RS \\ -(QR)(x) \int_1^x PS - (PS)(x) \int_1^x QR, \end{aligned}$$

so $F'(1) = 0$, thus $(x-1)^2 | F(x)$.

$$\begin{aligned} F''(x) &= (RS)' \int PQ + RS\, PQ + (PQ)' \int RS + PQ\, RS \\ &\quad -(QR)' \int PS - QR\, PS - (PS)' \int QR - PS\, QR \\ &= (RS)' \int_1^x PQ + (PQ)' \int_1^x RS - (QR)' \int_1^x PS - (PS)' \int_1^x QR, \end{aligned}$$

so $F''(1) = 0$, and $(x-1)^3 | F(x)$.

$$\begin{aligned} F'''(x) &= (RS)'' \int PQ + (RS)' PQ + (PQ)'' \int RS + (PQ)' RS \\ &\quad -(QR)'' \int PS - (QR)' PS - (PS)'' \int QR - (PS)' QR \\ &= (RS)'' \int PQ + (PQ)'' \int RS - (QR)'' \int PS - (PS)'' \int QR \\ &\quad +(RS)'(PQ) + (PQ)' RS - [(QR)'(PS) + (PS)' QR]. \end{aligned}$$

The terms without integral factors are $(PQRS)' - (PQRS)' = 0$ so $F'''(1) = 0$, and $(x-1)^4$ divides $F(x)$.

Note $P = y,\ Q = y,\ R = 3,\ S = 4$ gives

$$\begin{aligned} F(x) &= \int_1^x y^2 \int_1^x 12 - \int_1^x 3y \int_1^x 4y \\ &= 12\left(\frac{y^3}{3}\Big|_1^x \cdot y\Big|_1^x\right) - 12\left(\frac{y^2}{2}\Big|_1^x \cdot \frac{y^2}{2}\Big|_1^x\right) \\ &= 4(x^3-1)(x-1) - 3(x^2-1)(x^2-1) \\ &= (x-1)^2[4(x^2+x+1) - 3(x+1)^2] \\ &= (x-1)^2[4x^2+4x+4-3x^2-6x-3] \\ &= (x-1)^2[x^2-2x+1] \\ &= (x-1)^4 \end{aligned}$$

so $(x-1)^5 \nmid F(x)$ in general.

Also solved by:

Graduates: Tom Hunter (MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN)

One unacceptable solution was received.