PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2003 Series)

Problem: Determine the number of integers $n \geq 2$ for which the congruence $x^{25} \equiv x \pmod{n}$ is true for all integers x.

Solution (by Amit Shirsat, Gr. CS, edited by the Panel)

If for all $x \in \mathbb{N}$, $x^{25} \equiv x \pmod{mn}$ where $(m,n)=1$ then $x^{25} \equiv x \pmod{m}$ and $x^{25} \equiv x \pmod{n}$. Conversely, if $x^{25} \equiv x \pmod{m}$ and $x^{25} \equiv x \pmod{n}$ then $x^{25} \equiv x \pmod{mn}$, so need consider only primes p such that $x^{25} \equiv x \pmod{p^r}$ for some $r \geq 1$. If $x^{25} \equiv x \pmod{p^r}$ and $r > 1$ then $(p^{r-1})^{25} \equiv 0 \pmod{p^r}$, but $p^{r-1} \not\equiv p^r \pmod{p^r}$, so $r > 1$ is not possible so we need $x^{24} \equiv 1 \pmod{p}$. By Fermat's Little Theorem, $x^{24} \equiv 1 \pmod{p}$ if $(p-1)|24$, i.e. $p = 2, 3, 5, 7, 13$. By using $2^{25} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^2 \cdot 241$ one tests easily that the factor 241 does not work. Thus the numbers n sought are products of the numbers $2, 3, 5, 7, 13$ with at least one factor. There are $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 - 1 = 31$ such numbers.

Also solved by:

Undergraduates: Chad Aeschliman (Fr. Engr.)

Graduates: Anandateertha Mangasuli (MA)

Others: J.L.C. (Fishers, IN), Jeff Hammerbacher (Ft. Wayne, IN), Namig Mammadov (Baku, Azerbaijan), Alex Miller (St. Anthony H.S., MN), Regis J. Serinko (PhD, State Coll., PA)