PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2003 Series)

Problem: Define the numbers e_k by $e_0 = 0$, $e_k = \exp(e_{k-1})$ for $k \geq 1$. Determine the functions f_k for which

$$f_0(x) = x, \quad f_k' = \frac{1}{f_{k-1} f_{k-2} \cdots f_0} \quad \text{for } k \geq 1$$

on the interval $[e_k, \infty)$, and all $f_k(e_k) = 0$.

Solution (by Rob Pratt, Gr. U. North Carolina)

We show by induction that $f_k(x) = \ln^k x$, the k-fold composition of ln with itself. For $k = 0$, we have $f_0(x) = x = \ln^0 x$. Now assume that $f_k(x) = \ln^k x$ for some $k \geq 0$. Then

$$f_{k+1}'(x) = \frac{1}{f_k(x) f_{k-1}(x) \cdots f_0(x)} = \frac{f_k'(x)}{f_k(x)}.$$

So

$$f_{k+1}(x) = \int \frac{f_k'(x)}{f_k(x)} dx = \ln f_k(x) + C = \ln \ln^k x + C = \ln^{k+1} x + C$$

for some constant C. But

$$0 = f_{k+1}(e_{k+1}) = \ln^{k+1} e_{k+1} + C = C.$$

Hence $f_{k+1}(x) = \ln^{k+1} x$, establishing the induction.

Also solved by:

Undergraduates: Neel Mehta (Fr. AAE), M. M. Ahmad Zabidi (Fr. Biol)

Graduates: Tom Engelsman (ECE), Amit Shirsat (CS), Qi Xu (ChE)

Others: J.L.C. (Fishers, IN), Marcio A. A. Cohen (Brazil), Namig Mammadov (Baku, Azerbaijan)

Three correct solutions to problem 5 were misfiled and not corrected last week. They are for Marcio A. A. Cohen (Eng, Brazil), Steven Landy (Phys at IUPUI), Yifan Liang (Gr. ECE).