PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2003 Series)

Problem: Let $y(x)$ be a continuously differentiable real-valued function on $\mathbb{R}$. Show that, if $(y'(x))^2 + y^3(x) \to 0$ as $x \to +\infty$, then $y(x) \to 0$ as $x \to +\infty$.

Solution (by the Panel)

There are three cases to be considered.

1. Suppose $y(x)$ changes sign at x_n, $x_n \to \infty$. Then $y(x)$ has a maximum or minimum at ξ_n, $x_n < \xi_n < x_{n+1}$, $y'(\xi_n) = 0$, $|y(x)| \leq |y(\xi_n)|$ for $x_n \leq x \leq x_{n+1}$, $|y(\xi_n)| \to 0$, $\therefore y(x) \to 0$.
2. $y(x)$ does not change sign for $x \geq u$, say $y(x) \geq 0$ for $x \geq u$. Then $y(x) \to 0$ for $x \to \infty$.
3. $y(x) \leq 0$ for $x \geq u$. Set $z = -y$, then $(z')^2 - z^3 \to 0$ and since $(z'^2 - z^3)$ is arbitrarily small for sufficiently large x, $z(x)$ differs arbitrarily little, for sufficiently large x, from $w(x)$ for which $w'^2 - w^3 = 0$, or $(w' - w^{3/2})(w' + w^{3/2}) = 0$. If $w' + w^{3/2} \neq 0$ at some x, then $w' + w^{3/2} \neq 0$ on an interval I_1, so $w' - w^{3/2} = 0$ on I_1. If I is finite then there is an abutting interval I_2 on which $w' + w^{3/2} = 0 : w(x) = (-\frac{1}{2}x + c_1)^{-2}$ on I_1 and $w(x) = -(\frac{1}{2}x + c_2)^{-2}$ on I_2. Also $w'(x) = (-\frac{1}{2}x + c_1)^{-3}$ on I_1, $w'(x) = -(\frac{1}{2}x + c_2)^{-3}$ on I_2. At the point $x = a$ where I_1 and I_2 abut, we have $(-\frac{1}{2}a + c_1)^{-3} = -(\frac{1}{2}a + c_2)^{-3}$, hence $c_1 = -c_2$ and $w(x) = (\frac{1}{2}x + c)^{-2}$ on $I_1 \cup I_2$. It follows that $w(x) = (\frac{1}{2}x + c)^{-2}$ for all x, hence $w(x) \to 0$, $z(x) \to 0$, $y(x) \to 0$.

There was no correct solution, only two incorrect solutions.