PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2003 Series)

Problem: Find the exact sum of the series $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{5\cdot6\cdot7\cdot8}+\cdots$.

Solution (by Tom Engelsman, Grad ECE, modified by the Panel)

$$\frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)}=\int_0^1\int_0^x\int_0^{x_1}\int_0^{x_2}x_3^{4n}dx_3dx_2dx_1dx=\frac{1}{6}\int_0^1(1-x)^3x^{4n}dx.$$

Hence

$$S=\frac{1}{6}\int_0^1\frac{(1-x)^3}{1-x^4}dx=\frac{1}{6}\int_0^1\frac{(1-x)^2}{(1+x)(1+x^2)}dx$$

which decomposes to

$$\frac{1}{6}\int_0^1\left(-\frac{x+1}{1+x^2}+\frac{2}{1+x}\right)dy.$$

Integrating gives

$$\frac{1}{6}\left[2\ln(1+x)-\frac{1}{2}\ln(1+x^2)-\tan^{-1}x\right]_0^1$$

$$=\frac{1}{6}\left[2\ln2-\frac{1}{2}\ln2-\frac{\pi}{4}\right]=\frac{1}{6}\left(\frac{3}{2}\ln2-\frac{\pi}{4}\right)=\frac{1}{4}\ln2-\frac{\pi}{24}.$$

Also solved by:

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