PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2003 Series)

Problem: A point P is chosen at random with respect to the uniform distribution in an equilateral triangle T. What is the probability that there is a point Q in T whose distance from P is larger than the altitude of T? (The answer can be found without integration.)

Solution (by the Panel)

Let AOB be the vertices of T, M the midpoint of OB, C the orthocenter of T, and R the intersection of the altitude of AB (say of length h) and the circle with center A and radius $\overline{AM}$. The sought probability is

$$p = 6(\text{area}(ORM))/\frac{1}{4}\sqrt{3}$$

if 1 is the length of the side of T.

On taking origin at O, positive x-axis along OB, and positive y-axis through O in the direction of MA, the coordinates x, y of Q satisfy

$$y = \frac{x}{3}\sqrt{3}, \quad (x - \frac{1}{2})^2 + (y - \frac{1}{2}\sqrt{3})^2 = \frac{3}{4}.$$

One finds $x = \frac{3}{4} - \frac{1}{4}\sqrt{6}, y = \frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{2}$.

Now $|ORM| = |OCM| - |RCM| = |OCM| - (|RAM| - |RAC|)$,
where $|OCM| = \frac{1}{24}\sqrt{3}$, $|RAC| = \frac{1}{2}\frac{1}{3}\sqrt{3}(\frac{1}{2} - x) = \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3}$,
$|RAM| = \frac{h^2}{2}\sin^{-1}\left(\frac{\frac{1}{2}-x}{h}\right) = \frac{3}{8}\sin^{-1}\left(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}\right)$.
Hence

$$\begin{aligned}|ORM| &= \frac{1}{24}\sqrt{3} - \frac{3}{8}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}) + \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3}, \\ &= \frac{1}{8}\sqrt{2} - \frac{3}{8}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}),\end{aligned}$$

and so

$$p = \sqrt{6} - 3\sqrt{3}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}) = 0.2067.$$

Also solved by:

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One incorrect solution was received.