PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2003 Series)

Problem: A given triangle T contains a point P in its interior. Let A, B, C be points one on each side of T.

a) Show that $|PA| + |PB| + |PC| \geq h$, where h is the length of the shortest altitude of T.

b) Show that equality in the above is attained if and only if T is equilateral and PA, PB, PC are orthogonal to their respective sides.

Solution (Yifan Liang, Gr. ECE)

a) Assume $T = \triangle A_0B_0C_0, |A_0B_0| \geq |B_0C_0| \geq |C_0A_0|$. The area of T is

$$\begin{aligned} S_T &= \frac{1}{2}|A_0B_0| \cdot h \\ &= S_{\triangle A_0B_0P} + S_{\triangle B_0C_0P} + S_{\triangle C_0A_0P} \\ &\leq \frac{1}{2}|A_0B_0| \cdot |PC| + \frac{1}{2}|B_0C_0| \cdot |PA| + \frac{1}{2}|C_0A_0| \cdot |PB| \\ &\leq \frac{1}{2}|A_0B_0|(|PC| + |PA| + |PB|), \end{aligned}$$

hence $|PA| + |PB| + |PC| \geq h$.

b) The first equality above is attained if and only if

$$PA \perp B_0C_0,\ PB \perp A_0C_0\ PC \perp A_0B_0.$$

The second equality holds if and only if

$$|A_0B_0| = |B_0C_0| = |C_0A_0|.$$

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