PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2003 Series)

Problem: Determine the complex-valued functions $f(x)$ which have power series expansions that converge near 0 and which satisfy $2f^2(x) - f(2x) = 1$ inside the circle of convergence.

Solution (by The Panel)

We are given that $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$. From $2(f(0))^2 - f(0) - 1 = 0$ we find that $f(0)$ can be $-\frac{1}{2}$ or 1.

By mathematical induction we can show:

$$(1) \qquad \frac{d^m}{dx^m}[g(x) \cdot h(x)] = \sum_{k=0}^{m} \binom{m}{k} g^{(k)}(x) h^{(m-k)}(x).$$

Taking the m'th derivative of $2(f(x))^2 - f(2x) - 1 = 0$ for $m \geq 1$

$$2\sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(x) - 2^m f^{(m)}(x) = 0.$$

For $x = 0$, $4f(0)f^{(m)}(0) + 2\sum_{k=1}^{m-1} f^{(k)}(0)f^{(m-k)}(0) - 2^m f^{(m)}(x) = 0$.

This gives a recursion formula for $f^{(m)}(0)$.

$$(2) \qquad (2^m - 4f(0))f^{(m)}(0) = 2\sum_{k=1}^{m-1} \binom{m}{k} f^{(k)}(0) f^{m-k}(0).$$

For $m = 1$ the sum on the right side vanishes, being void, so $(2 - 4f(0)|f'(0) = 0$ which, for either value of $f(0)$ gives $f'(0) = 0$.

For $m = 2$, $(4 - 4f(0))f''(0) = 2(f'(0))^2 = 0$. For $f(0) = -\frac{1}{2}$ this gives $f''(0) = 0$ but for $f(0) = 1$, $f''(0)$ can be any number, call it d.

Case 1 $f(0) = -\frac{1}{2}$. From (2) $(2^m + 2)f^{(m)}(0) = 2\sum_{k=1}^{m-1} \binom{m}{k} f^{(k)}(0) f^{(m-k)}(0)$. If $f^{(k)}(0) = 0$ for $k = 1, 2, \cdots (m-1)$ (when $m \geq 1$), the right side of (2) is zero so $f^{(m)}(0) = 0$. Since $f'(0) = 0$ mathematical induction shows that $f^m(0) = 0$ for $m \geq 1$. Thus $f(x) = f(0) = -\frac{1}{2}$.

Case 2 $f(0) = 1$. The right side of (2) for $m = 3$ is $2 \cdot 3(f'(0)f''(0) + f''(0)f'(0)) = 0$. Thus $(2^3 - 4)f^{(3)}(0) = 0$ giving $f^{(3)}(0) = 0$. If $f^{(k)}(0) = 0$ $k = 1, 3, \cdots, 2r-1$, then

$(2^{2r+1}-4)f^{(2r+1)}(0) = 2\sum_{k=1}^{2r}\binom{2r+l}{k}f^{(k)}(0)f^{(2r+1-k)}(0)$. On the right each produce has an odd and an even derivative and is, thus, 0. This shows $f^{(2r+1)}(0) = 0$ for all $r \geq 0$.

Writing out (2) for $m = 2r$

$$2(2^{2(r-1)}-1)f^{(2r)}(0) = \sum_{k=1}^{m-1}\binom{2r}{k}f^{(k)}(0).$$

The odd numbered terms on the right have products of odd derivatives and therefore vanish leaving

$$2(2^{2(r-1)}-1)f^{(2r)}(0) = \sum_{j=1}^{r-1}\binom{2r}{2j}f^{(2j)}(0)f^{(2r-2j)}(0).$$

For $r = 2$ we calculate $f^{(4)}(0) = d^2$ nd for $r = 3$, $f^{(6)}(0) = d^3$. With the induction hypothesis that $f^{(2k)}(0) = d^k$ for $k = 1, 2, \cdots, r-1$, $2(2^{2(r-1)}-1)f^{2r}(0) = [\sum_{j=1}^{r-1}\binom{2r}{2j}]d^r$.

The sum on the right may be found by evaluating $\frac{1}{2}[(1+t)^{2r}-(1-t)^{2r}]$ at $t = 1$. From this mathematical induction shows $f^{(2r)}(0) = d^r[2^{2r} - \binom{2r}{0} - \binom{2r}{2r}]/2(2^{2(r-1)}-1) = d^r$ which gives

$$f(x) = \sum_{k=0}^{\infty}\frac{d^k}{(2k)!}x^{2k} = \cosh(\sqrt{d}x).$$

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