PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2004 Series)

Problem: Show that $1 - \frac{x}{3} < \frac{\sin x}{x} < 1.1 - \frac{x}{4}$ for $0 < x \leq \pi$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

a) Let

$$\begin{aligned} f(x) &= \sin x - x + \frac{x^2}{3}, \quad \text{so} \\ f'(x) &= \cos x - 1 + \frac{2x}{3} \quad \text{and} \\ f''(x) &= -\sin x + \frac{2}{3}. \end{aligned}$$

$f''(x) > 0$ except for $a < x < b$ where $\sin a = \sin b = \frac{2}{3}$ and $a < \frac{\pi}{2} < b$. Thus $f'(x)$ increases from 0 to a maximum at $x = a$, then decreases to a minimum at $x = b$, and then increases from b to π. The minimum value is $f'(b) = \cos b - 1 + \frac{2b}{3} > \frac{(\sqrt{5})}{3} - 1 + \frac{\pi}{3} > 0$. From this $f'(x) \geq 0$ for $0 \leq x \leq \pi$ and consequently $f(x)$ increases from 0 to $\pi(\frac{\pi}{3} - 1)$ and $\sin x \geq x - \frac{x^2}{3}$.

b) Let

$$\begin{aligned} g(x) &= \sin x - 1.1x + \frac{x^2}{4}, \quad \text{so} \\ g'(x) &= \cos x - 1.1 + \frac{x}{2} \quad \text{and} \\ g''(x) &= -\sin x + \frac{1}{2}. \end{aligned}$$

As in a) $g''(x) \geq 0$ except for $\frac{\pi}{6} < x < \frac{5\pi}{6}$ where $g''(x) < 0$. So $g'(x)$ increases from a value of -1.1 to a maximum of $\cos\frac{\pi}{6} - 1.1 + \frac{\pi}{12} > 0$. It must be zero at a point, $x = \alpha (0 \leq \alpha \leq \frac{\pi}{6})$. Similarly $g'(\beta) = 0$ for $\frac{\pi}{6} \leq \beta \leq \frac{5\pi}{6}$. Note that $g'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - 1.1 + \frac{\pi}{8} > 0$ and $g'(\frac{3\pi}{4}) = \frac{-\sqrt{2}}{2} - 1.1 + \frac{3\pi}{8} < 0$, so $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$. Then from 0, $g(x)$ decreases to a minimum at $x = \alpha$, then increases to a maximum value at $x = \beta$; $g'(\beta) = 0 = \cos\beta - 1.1 + \frac{\beta}{2}$, or $\beta = 2(1.1 - \cos\beta)$. Thus $g(\beta) = \sin\beta - 1.1\beta + \frac{\beta^2}{4} = \sin\beta + \cos^2\beta - (1.1)^2 = (.03 - \sin\beta)(\sin\beta - 0.7) < 0$ since $\sin\beta \geq \frac{\sqrt{2}}{2}$. From this $g(x) = \sin x - 1.1 + \frac{x^2}{4} \leq 0$ for $0 \leq x \leq \pi$.

Also solved by:

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