PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2004 Series)

Problem: Prove the identity $\sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N} \binom{N-k+3}{3} f(k)$.

Solution (by A. Plaza & M.A. Padron, Faculty ULPGC, Spain)

The proof is divided in three parts:

(1) $$\sum_{\ell=0}^{N} \sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N} \binom{N-k+1}{1} f(k).$$

This can be easily proved by induction:
For $N = 0$, we get $f(0) = f(0)$. Suppose that equation (1) holds for $N-1$:

$$\sum_{\ell=0}^{N-1} \sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N-1} \binom{N-k}{1} f(k).$$

Then, for index N we obtain:

$$\sum_{\ell=0}^{N} \sum_{k=0}^{\ell} f(k) = \sum_{\ell=0}^{N-1} \sum_{k=0}^{\ell} f(k) + \sum_{k=0}^{N} f(k) = \sum_{k=0}^{N-1} \binom{N-k}{1} f(k) + (N+1) f(k) = \sum_{k=0}^{N} \binom{N-k+1}{1} f(k).$$

Second, based on equation (1), we get:

(2) $$\sum_{m=0}^{N} \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} f(k) = \sum_{\ell=0}^{N} \binom{N-\ell+1}{1} \cdot \sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N} \binom{N-k+2}{2} f(k).$$

Third:

(3) $$\sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{\ell=0}^{m} \sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N} \binom{N-k+3}{3} f(k).$$

NOTE: The equation of the problem may be extended to a finite number of nested sums.

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