PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2004 Series)

Problem: Suppose $a_1, a_2, \cdots, a_n$ are real numbers and $|a_n| > \sum_{k=1}^{n-1} |a_k|$. Show that $f(x) = \sum_{k=1}^{n} a_k \cos kx$ has at least n zeros for $0 \le x \le \pi$.

Solution (by Qi Xu, graduate ChE)

Divide the interval $0 \le x \le \pi$ into n subintervals of the same size,

$$\left(\frac{k\pi}{n} \le x \le \frac{(k+1)\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

In each interval we have at one end $\cos nx = 1$ and at the other end $\cos nx = -1$. Hence, since

$$-\sum_{k=1}^{n-1} |a_k| \le \sum_{k=1}^{n-1} a_k \cos kx \le \sum_{k=1}^{n-1} |a_k|,$$

therefore, at one end,

$$f(x) = |a_n| + \sum_{k=1}^{n-1} a_k \cos kx \ge |a_n| - \sum_{k=1}^{n-1} |a_k| > 0,$$

while at the other end,

$$f(x) = -|a_n| + \sum_{k=1}^{n-1} a_k \cos kx \le -|a_n| + \sum_{k=1}^{n-1} |a_k| < 0.$$

Since $f(x)$ is continuous, there exists at least one x in this interval such that $f(x) = 0$. There are n such intervals, therefore, there are at least n zeros for $0 \le x \le \pi$.

Also solved by:

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One incorrect solution was received.