PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2004 Series)

Problem: Show that $S_n = 2^{1-n} \sum_{k\geq 0} \binom{n}{2k+1}(-3)^k$ may equal only $0, 1$, or -1. For which values of n is it equal to 0? to 1? to -1?

Solution (by Huai-Tzu You, Grad Aero & Astro)

Let $m = 2k+1$, then $k = \frac{m-1}{2}$, and we have

$$S_n = 2^{1-n} \sum_{m=\text{odd}} \binom{n}{m} (-3)^{\frac{m-1}{2}} = \frac{2^{1-n}}{\sqrt{-3}} \sum_{m=\text{odd}} \binom{n}{m} (\sqrt{-3})^m.$$

Also

$$\sum_{m=\text{odd}} \binom{n}{m} (\sqrt{-3})^m = \frac{1}{2}\{(1+\sqrt{-3})^n - (1-\sqrt{-3})^n\},$$

whence

$$S_n = \frac{1}{2^n\sqrt{3}i}\{(1+i\sqrt{3})^n - (1-i\sqrt{3})^n\}.$$

But

$$1 + i\sqrt{3} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2e^{i\frac{\pi}{3}},$$

hence

$$S_n = \frac{1}{\sqrt{3}i}\{e^{i\frac{n\pi}{3}} - e^{-i\frac{n\pi}{3}}\} = \frac{2}{\sqrt{3}}\sin(\frac{n\pi}{3}) = \frac{\sin(\frac{n\pi}{3})}{\sin(\frac{\pi}{3})},$$

i.e.

$$S_n = \begin{cases} 1 & n \equiv 1 \text{ or } 2 \pmod 6 \\ 0 & n \equiv 0 \text{ or } 3 \pmod 6 \\ -1 & n \equiv 4 \text{ or } 5 \pmod 6. \end{cases}$$

Also solved by:

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