PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2005 Series)

Problem: Show that the circumference of any rhombus (diamond) circumscribed about a given ellipse is no smaller that the circumference of the circumscribed rectangle about the same ellipse with sides parallel to the axes of the ellipse.

You may assume that the diagonals of the rhombus lie along the axes of the ellipse.

Solution (by Steven Landy, IUPUI Physics staff; edited by the Panel)

Let our ellipse be

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{1}$$

Suppose $\frac{1}{4}$ of the rhombus is along the line

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{2}$$

We wish to show that

$$A + B \le \sqrt{a^2 + b^2}, \tag{3}$$

when the line is tangent to the ellipse.
Solving (1) and (2) simultaneously shows there is a single root if and only if

$$1 = \frac{A^2}{a^2} + \frac{B^2}{b^2}.$$

In particular, $A \le a$, $B \le b$.
Then we can set $A = a\cos\alpha$, $B = b\sin\alpha$ with some α.
Then

$$A + B = a\cos\alpha + b\sin\alpha = \sqrt{a^2+b^2}\ \sin(\alpha + \phi) \tag{4}$$

where $\phi = \sin^{-1} \frac{a}{\sqrt{a^2+b^2}}$. Equation (4) shows that (3) is true.

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