PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2005 Series)

Problem:

(a) Show that the number of ways in which an odd positive integer n can be written as a sum of two or more consecutive positive integers is equal to the number of divisors d of $2n$ such that $1 < d < \sqrt{2n}$.

(b) Give all these sums for $n = 45$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

Notice first, that each such sum is uniquely determined by the number d of its terms.

(a) (i) Let's show first that if n can be written as a sum of d $(d > 1)$ consecutive positive integers then d is divisor of $2n$ and $1 < d < \sqrt{2n}$.
Indeed $n = m + (m+1) + \cdots + (m+d-1)$ where $m > 0$
$\iff n = md + \dfrac{d(d-1)}{2} \iff 2n = d(2m+d-1) \Rightarrow d$ is a divisor of $2n$ but

$$\begin{aligned} 2m-1 > 0 \Rightarrow d^2 &< d(2m+d-1) = 2n \\ \Rightarrow d &< \sqrt{2n} \end{aligned}$$

(ii) Let's show now that if d is a divisor of $2n$ and $1 < d < \sqrt{2n}$ then there is a unique $m > 0$ so that $n = m + (m+1) + \cdots + (m+d-1)$.
Indeed $2n = d \times k$ but $d < \sqrt{2n} \Rightarrow k > \sqrt{2n} > d$

$$2n = d \times k = d(2m+d-1) \Rightarrow 2m = k-(d-1)$$

but n is odd $\Rightarrow d$ and k don't have the same parity
$\Rightarrow (d-1)$ and k have the same parity
$\Rightarrow \quad m = \frac{k-(d-1)}{2} > 0$ is an integer, and is unique.
Consequently the 2 sets have the same number of elements.

(b) $n = 45$ the divisors of $2n = 90$ d and $1 < d \leq 9$ are $d = 2, 3, 5, 6, 9$.

$$\begin{aligned} 45 &= 22 + 23 \quad (m = 22, \quad d = 2) \\ &= 14 + 15 + 16 \quad (m = 14, \quad d = 3) \\ &= 7 + 8 + 9 + 10 + 11 \quad (m = 7, \quad d = 5) \\ &= 5 + 6 + 7 + 8 + 9 + 10 \quad (m = 5, \quad d = 6) \\ &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \quad (m = 1, \quad d = 9) \end{aligned}$$

Also solved by:

Undergraduates: Jason Anema (Sr. MA)

Others: Aaditya Muthukumaran (Chennai, India), A. Plaza (ULPGC, Spain), Steve Spindler