PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2005 Series)

Problem: Let h_1, h_2, h_3 be the altitudes of a triangle, and let ρ be the radius of its inscribed circle. Find the minimum of

$$\frac{h_1 + h_2 + h_3}{\rho}$$

over all triangles.

Solution (by Daniel Vacaru, Pitesti, Romania; edited by the Panel)

Let S be the area, and a, b, c, be the sides. We have $h_1 = \dfrac{2S}{a}$, $h_2 = \dfrac{2S}{b}$, $h_3 = \dfrac{2S}{c}$. Also, we have $\rho = \dfrac{S}{p}$, where $2p = a + b + c$. Therefore,

$$\frac{h_1 + h_2 + h_3}{\rho} = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}.$$

Then, since $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$, we have $\dfrac{a}{b} + \dfrac{b}{a} + \dfrac{a}{c} + \dfrac{c}{a} + \dfrac{b}{c} + \dfrac{c}{b} \geq 6$.

Therefore, the minimum is 9. The triangle for which the minimum is attained is the equilateral triangle.

Also solved by:

Undergraduates: Alan Bernstein

Graduates: Ashish Rao (ECE)

Others: Prithwijit De (Ireland), Georges Ghosn (Quebec), Steven Landy (IUPUI), M. Rappaport (Worcester Yeshiva Acad.), Gabriel Vrinceanu (Bucharest)