PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2005 Series)

Problem: Suppose $\left\{a_n\right\}_{n=1}^{\infty}$ be recursively defined by $a_0 > 1$, $a_1 > 0$, $a_2 > 0$,

$$a_{n+3} = \frac{1 + a_{n+1} + a_{n+2}}{a_n}, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Show that a_n has period 8, i. e.

$$a_{n+8} = a_n \quad \text{for any} \quad n \geq 0.$$

Solution (by the Panel)

Subtract the equations

$$a_{n+3}\, a_n = 1 + a_{n+1} + a_{n+2}$$

$$a_{n+2}\, a_{n-1} = 1 + a_n + a_{n+1}$$

to get

$$a_{n+3}\, a_n - a_{n+2}\, a_{n-1} = a_{n+2} - a_n,$$

or

$$a_n(a_{n+3} + 1) = a_{n+2}(a_{n-1} + 1).$$

Add $a_n a_{n+2}$ to both sides to get

$$a_n(1 + a_{n+2} + a_{n+3}) = a_{n+2}(1 + a_{n-1} + a_n).$$

Using the recursive equation, we get

$$a_n\, a_{n+1}\, a_{n+4} = a_{n+2}\, a_{n+1}\, a_{n-2}.$$

Since all terms are positive, we cancel a_{n+1} to obtain

$$a_{n-2}\, a_{n+2} = a_n\, a_{n+4}.$$

Replace n by $n - 2$ to get

$$a_n\, a_{n-4} = a_{n+2}\, a_{n-2}.$$

The last two identities imply $a_n a_{n-4} = a_n a_{n+4}$, therefore

$$a_{n-4} = a_{n+4}, \quad n \geq 4 \quad \Rightarrow \quad a_n = a_{n+8}, \quad n \geq 0.$$

Also solved by:

Undergraduates: Alan Bernstein, Justin Woo (Jr. ECE)

Graduates: Tom Engelsman, Niru Kumari (ME)

Others: Georges Ghosn (Quebec), Byungsoo Kim (Seoul Natl. Univ.), Jeff Ledford (Gainesville, GA), A. Plaza (ULPGC, Spain), Steve Spindler, Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

Update on Problem No. 4:

That problem was solved also by Justin Woo (Jr. ECE).