PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2005 Series)

Problem: Let Q be a non–degenerate convex quadrilateral inscribed in a circle. Show that the four lines, each passing through the midpoint at a side of Q and perpendicular to the opposite side, meet in a point.

Solution (by the Panel)

Let the origin of the coordinate system be at the center of the circle, and let the vertices of Q be at the vectors a, b, c, d. Then $|a| = |b| = |c| = |d| = 1$, and

$$(a - b) \cdot (a + b) = 0,$$

similarly for any other pair. Therefore, $a + b$ is perpendicular to the side (a, b);
$b+c$ is perpendicular to (b, c), etc. The equation of the line through the midpoint of (a, b), perpendicular to (c, d) is

$$x(s_1) = \frac{a+b}{2} + s_1(c + d), \qquad -\infty < s < \infty.$$

We apply the same argument to each pair of consecutive vertices. As a result, we reduce the problem to the following one: Show that there is unique solution (s_1, s_2, s_3, s_4, x) of the system:

$$\begin{aligned} x &= \frac{1}{2}(a + b) + s_1(c + d), \\ x &= \frac{1}{2}(b + c) + s_2(d + a), \\ x &= \frac{1}{2}(c + d) + s_3(a + b), \\ x &= \frac{1}{2}(d + a) + s_4(b + c). \end{aligned}$$

An obvious solution of this system is $s_1 = s_2 = s_3 = s_4 = \frac{1}{2}$, and

$$x = \frac{1}{2}(a + b + c + d).$$

This solution is unique, because the first two lines, for example, are not parallel, so they have unique common point.

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