PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2005 Series)

Problem: Let $P(x)$ be a polynomial such that all roots of $P(x)$ are real.

(a) Prove that

$$\Big(P'(x)\Big)^2 \geq P(x)P''(x) \qquad \text{for all real } x.$$

(b) For what x does an equality hold?

Solution (by Georges Ghosn, Quebec)

The problem is trivial if $\deg P = 0$, so we assume $\deg P \neq 0$. Since all roots of $P(x)$ are real, $P(x)$ can be expressed as:

$$P(x) = A(x-x_1)^{m_1}(x-x_2)^{m_2}\dots(x-x_n)^{m_n} \text{ with } m_i \geq 1, \text{ and } \deg P = m_1 + \cdots + m_n.$$

(a) This inequality is satisfied if $x = x_i$, so we suppose $x \neq x_i$. Then

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{m_i}{x - x_i}, \qquad \forall x \in \mathbb{R} \quad (x \neq x_i).$$

Taking the derivative of both terms of this equality yield to:

$$\begin{aligned}
\frac{P''(x)P(x) - P'^2(x)}{P^2(x)} &= \sum_{i=1}^{n} \frac{-m_i}{(x-x_i)^2} \\
&\Rightarrow P'^2(x) - P(x)P''(x) \\
&= \left(\sum_{i=1}^{n} \frac{m_i}{(x-x_i)^2}\right)P^2(x) \geq 0 \\
&\Rightarrow \quad P'^2(x) \geq P(x)P''(x) \quad \forall x \in \mathbb{R}.
\end{aligned}$$

(b) Since we have from (a),

$$P'^2(x) - P(x)P''(x) > 0 \quad \forall x \in \mathbb{R} \quad x \neq x_i$$

and

$$P'^2(x_i) - P(x_i)P''(x_i) = P'^2(x_i)$$

The equality holds only for all roots of $P(x)$ of multiplicity equal 2 or higher, or for polynomials of degree zero (constants) for all x.

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