PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2005 Series)

Problem: Let $P_1, P_2, \ldots, P_n$ be points on a sphere of radius 1 in $\mathbb{R}^3$. Let d_{ij} be the distance (in $\mathbb{R}^3$) from P_i to P_j.

(a) Prove that $\sum_{i<j} d_{ij}^2 \leq n^2$.

(b) When is $\sum d_{ij}^2 = n^2$?

Solution (by Georges Ghosn, Quebec)

The coordinates of $P_i (i = 1 \ldots n)$ in a cartesian coordinate system having its origin at the sphere center verify $x_i^2 + y_i^2 + z_i^2 = 1$.

(a) $d_{ij}^2 = P_iP_j^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 2 - 2x_ix_j - 2y_iy_j - 2z_iz_j$
but $P_iP_j^2 = P_jP_i^2$ and $P_iP_i^2 = 0$, therefore

$$\sum_{i<j} d_{ij}^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(1 - x_ix_j - y_iy_j - z_iz_j)$$
$$= n^2 - \left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{i=1}^{n} y_i\right)^2 - \left(\sum_{i=1}^{n} z_i\right)^2 \leq n^2$$

(b) The equality holds if and only if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i = 0$. That's mean that the center of mass of $P_1, \ldots P_n$ is at the origin.

Also solved by:

<u>Graduates</u>: Niru Kumari (ME)

<u>Others</u>: Prasad Chebulu (CMU, Pittsburg), Andrew Ferguson (Scotland), Steven Landy (IUPUI Physics staff), Joe Underbrink (IUPUI) Daniel Vacaru (Pitesti, Romania)