PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2005 Series)

Problem: Let $f(x)$ be a real–valued function on the open interval (0,1). Show that if

$$\lim_{x\to 0} f(x) = 0, \quad \text{and} \quad \lim_{x\to 0} \frac{f(x) - f(x/2)}{x} = 0,$$

then

$$\lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Solution (by Georges Ghosn, Quebec)

$\displaystyle\lim_{x\to 0} \frac{f(x) - f(x/2)}{x} = 0 \Leftrightarrow \forall\varepsilon > 0,\ \exists\alpha > 0 \quad / \quad 0 < x < \alpha \Rightarrow \left| f(x) - f\left(\frac{x}{2}\right)\right| < \varepsilon x.$

For any integer $n \geq 1$, we have $\displaystyle 0 < \frac{x}{2^{n-1}} \leq x < \alpha \Rightarrow \left| f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right)\right| < \varepsilon\frac{x}{2^{n-1}}.$

Adding all these inequalities yields to:

$$\begin{aligned}\left| f(x) - f\left(\frac{x}{2^n}\right)\right| &\leq \left| f(x) - f\left(\frac{x}{2}\right)\right| + \cdots + \left| f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right)\right| \\ &< \varepsilon x\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) = 2\varepsilon x\left(1 - \frac{1}{2^n}\right) < 2\varepsilon x.\end{aligned}$$

Hence $\displaystyle\left| f(x) - f\left(\frac{x}{2^n}\right)\right| < 2\varepsilon x$ is satisfied for every positive integer n, and $0 < x < \alpha$.

On the other hand,

$$\begin{aligned}&\lim_{x\to 0}\left| f(x) - f\left(\frac{x}{2^n}\right)\right| = |f(x)| \quad \left(\text{because } \lim_{x\to 0} f(x) = 0\right) \\ &\Rightarrow |f(x)| \leq 2\varepsilon x \quad \text{for} \quad 0 < x < \alpha, \\ &\Rightarrow \lim_{x\to 0}\frac{f(x)}{x} = 0.\end{aligned}$$

All attempts to solve the problem using L'Hôpital Rule or Taylor expansions are incorrect because, among other reasons, we do not know that f is differentiable.

Also, at least partially solved by:

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