PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2006 Series)

Problem: Let ε be an ellipse, which is not a circle, with center O. Let $P \in \varepsilon$ be a point at which the angle between the tangent to ε at P and $\overrightarrow{OP}$ is minimal. Find the angle that $\overrightarrow{OP}$ makes with the major axis.

Solution (by Georges Ghosn, Quebec; edited by the Panel)

The ellipse equation is $\varepsilon : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and by symmetry we can assume that $P(x_1, y_1) \in \varepsilon$ is in the first quadrant. The equation of the tangent to ε at P is $\dfrac{Xx_1}{a^2} + \dfrac{Yy_1}{b^2} = 1$. Therefore the slope of the tangent is $m_1 = -\dfrac{b^2 x_1}{a^2 y_1}$ and the slope of $\overrightarrow{OP}$ is $m_2 = \dfrac{y_1}{x_1}$. Therefore the tangent of the angle between the above lines is

$$\frac{m_2 - m_1}{1 + m_2 m_1} = \frac{a^2 y_1^2 + b^2 x_1^2}{(a^2 - b^2) x_1 y_1} = \frac{a^2 b^2}{(a^2 - b^2) x_1 y_1}.$$

This angle is minimal if and only if this tangent is minimal and therefore $x_1 y_1$ is maximal. But $\dfrac{x_1}{a} \cdot \dfrac{y_1}{b} \leq \dfrac{1}{2}\left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}\right) = \dfrac{1}{2} \Rightarrow \quad x_1 y_1 \leq \dfrac{ab}{2}$. Therefore $x_1 y_1$ is maximal iff $\dfrac{x_1}{a} = \dfrac{y_1}{b} = \dfrac{\sqrt{2}}{2}$. Finally the angle that $\overrightarrow{OP}$ makes with the major axis is $\tan^{-1} \dfrac{y_1}{x_1} = \tan^{-1}\left(\dfrac{b}{a}\right)$.

At least partially solved by:

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